THE JOURNAL

OF THE

INDIAN MATHEMATICAL SOCIETY

2047

555

NEW SERIES

Volume V *No. 4.* *December 1941*

OFFICE OF THE REGISTRAR OF ... 21 SEP 1942 MADRAS

CONTENTS

	Pages.
F. W. Levi: On the Number of Generators of a free Product, and a Lemma of Alexander Kurosch ..	149—155
C. Racine: Contribution to the Relativistic Problem of n Bodies (I) ..	156—164
——: Contribution to the Relativistic Problem of n Bodies (II) ..	165—178
S. M. Shah: A Theorem on Integral Functions of Integral order ..	179—188
——: Note on a Theorem of Polya ..	189—191
Hans Raj Gupta: On Numbers of the Form $4^a(8b+7)$ ..	192—202
ERRATA	
Table of Contents for Volume V	

-600 00

MADRAS:
PRINTED AT THE MADRAS LAW JOURNAL PRESS, MYLAPORE.

The Indian Mathematical Society

Founded in 1907

President: R. VAIDYANATHASWAMI, M.A., D.SC. (University of Madras.)

Treasurer: L. N. SUBRAMANIAM, M.A., (Madras Christian College, Madras.)

Secretaries: S. MAHADEVA IYER, M.A., L.T., (Presidency College, Madras.)

RAM BEHARI, M.A., PH.D., (St. Stephen's College, Delhi.)

The Committee of Management includes the President, the Secretaries, the Treasurer and the following members:

R. P. SHINTRE (Librarian), D. D. KOSAMBI, F. W. LEVI, A. NARASINGA RAO, A. N. SINGH, P. L. SRIVASTAVA, M. R. SIDDIQUI and J. MACLEAN.

The Society publishes two quarterlies, the *Journal of the Indian Mathematical Society*, and the *Mathematics Student*, which are published in March, June, September and December. The *Journal* is managed by the following Board of Editors.

R. VAIDYANATHASWAMI (*Chairman*)

K. ANANDA RAO — C. RACINE
D. D. KOSAMBI — N. R. SEN
F. W. LEVI — M. R. SIDDIQUI
S. S. PILLAI — T. VIJAYARAGHAVAN

C. N. SRINIVASIENGAR (*Secretary*)

1. Papers intended for the *Journal* should be addressed to C. N. Srinivasiengar, D.SC., College of Engineering, Bangalore, and papers intended for the *Student* should be addressed to A. Narasinga Rao, M.A., D.SC., Annamalai University, Annamalai Nagar.

2. Authors of papers appearing in the *Journal* will receive 50 reprints free of charge. Extra reprints can be had, at cost price, on intimation.

3. The Secretary should be addressed to for back numbers of both the Quarterlies, as well as by intending members and subscribers.

4. **Subscription for each of the Quarterlies is Rs. 6 per year. Single copy Rs. 1-8-0.**

129113

20247

ON THE NUMBER OF GENERATORS OF A FREE PRODUCT, AND A LEMMA OF ALEXANDER KUROSCH.

BY

F. W. LEVI, *Calcutta University*

[Received 30 June 1941]

Let a group G be generated by elements $a_1, \ldots, a_m$ which are supposed to be $\neq 1$, but not by a smaller number of elements of G, then this *minimum number of generators* will be denoted by

$$m(G). \tag{1}$$

If the order of G is equal to one, put $m(G) = 0$. The invariant $m(G)$ of a group G has not yet been much investigated. One may suppose that the minimum number of generators of a free product is equal to the sum of the corresponding numbers of the free factors, but this supposition (which is true for large classes of groups) has neither been confirmed by a general proof, nor been refuted by a counter-example.

In this paper, the supposition will be proved for the simplest non-trivial case, namely when $m(A*B) = 2$. This statement suffices to establish in a simple way a theorem, an interesting special case of which, Prof. Alexander Kurosch* has already proved by the help of very intricate investigations.

1. Let N be a normal subgroup of G, then G/N is homomorphic to G, and the homomorphism maps any set of generators of G on a set of generators of G/N.

* Alexander Kurosch; "Zum Zerlegungsproblem der Theorie der freien Produkte," *Receuille Mathematique* (nouvelle série) 2(44) (1937), 995-1001.

JL

Bma1,N3

Hence

$$m(G/N) \leqslant m(G). \tag{2}$$

If in particular, G is a free product, say $G = A*B$, then A and B are isomorphic to factorgroups of $A*B$. Hence

$$m(A) \leqslant m(A*B) \leqslant m(B). \tag{3}$$

On the other hand, a set of generators of A and a set of generators of B form together a set of generators of $A*B$. Hence

$$m(A*B) \leqslant m(A)+m(B). \tag{4}$$

If the order of a group is a non-enumerable infinity, then this order is equal to the number of any set of different generators. From this, one sees easily that if the group $A*B$ contains a non-enumerable infinity of elements, the symbol $\leqslant$ in the inequality (4) can be replaced by $=$. The corresponding holds if at least one of the numbers $m(A)$, $m(B)$, $m(A*B)$ denotes an enumerable infinity. Hence inequality cannot occur unless the three numbers are finite. If in particular $m(A)=0$, then $A*B=B$ and therefore equality holds in (4). Similarly, when $A*B$ is a free group; in this case, the minimum number of generators is equal to the number of free generators*, wherefrom the equation follows immediately.

2. It will be proved that if $m(A*B)=2$, the equation

$$m(A*B)=m(A)+m(B)$$

holds. In other words:

THEOREM. *If A and B are groups, each containing more than one element, and $m(A*B)=2$, then A and B are cyclic groups.*

PROOF. The letter a (with or without an index) denotes elements $\neq 1$ of A; similarly the letter b denotes elements $\neq 1$ of B. The groups A and B are supposed to have only the element 1 in common; thus a free product $G=A*B$ can be constructed. The elements $\neq 1$ of G can be

* F. W. Levi: *Über die Untergruppen der freien Gruppen* (II), *Math. Zeit.* 37 (1933), 90-7.

OFFICE OF ... 21 SEP 1942

represented as alternating (a, b)-products, and this representation is unique. As usual, the number of factors is called the *length* of the element. By the supposition of the theorem, G can be generated by two elements, say α_1 and α_2. If both the generators are of the length 1, one of them is an element a and the other is an element b. In this case, A is the cyclic group generated by a, and B is the cyclic group generated by b, thus the condition of the theorem is satisfied. To prove the theorem, it is therefore necessary and sufficient to show that the generators of G can be selected in such a way that both are of the length 1. To show this, it will be supposed that the generators α_1, α_2 are chosen in such a manner that α_1 has the smallest possible length, say n_1, and that α_2 has the smallest length, say n_2, which is compatible with the fact that α_1 has the length n_1. It will be proved that $n_1 = n_2 = 1$.

If α_1, α_2 are any pair of generators, then we get other pairs of generators by putting either

$$\beta_i = c\alpha_i c^{-1},\ i = 1, 2 \quad ; \tag{5}$$

or

$$\gamma_1 = (\alpha_1^{\pm 1}\alpha_2^{m})^{\pm 1},\ \gamma_2 = \alpha_2^{\pm 1} \quad ; \tag{6}$$

or

$$\delta_1 = \alpha_1^{\pm 1},\ \delta_2 = (\alpha_2^{\pm 1}\alpha_1^{m})^{\pm 1}, \tag{7}$$

where c is any element of G, the symbols ± 1 are independent of each other and m is any integral number. If α_1, α_2 have the minimum conditions as supposed above, then we have

$$\text{length } \beta_1 \geqslant n_1,\ \text{length } \gamma_1 \geqslant n_1,\ \text{length } \delta_2 \geqslant n_2. \tag{8}$$

As α_1 and α_2 generate G, every element d of G can be expressed as an α-product. We get the alternating (a, b)-product which represents d if we put into the α-product the (a, b)-products corresponding to α_1 and α_2, amalgamate products of consecutive a-elements and of consecutive b-elements, strike out the element 1 wherever it occurs, and repeat this procedure until the a-elements and b-elements are alternating. In particular, every element of A and

every element of B can be generated in this manner. The process of amalgamation can start from such places only where in the α-product, α-factors are meeting, and the last (a, b)-element of the first factor is of the same kind as the first element of the second factor (both a's or both b's). To study the process more closely, the following graphic representation of (a, b)-products is useful. An (a, b)-product of length $2k+1$ (where $k \geqslant 0$) is represented by an isosceles right-angled triangle whose base is placed on the line. The factors a and b are placed in their correct order and equidistant along the other two sides. Thus the first and last elements have the *height* 1, the second and the penultimate elements have the height 2, etc., the central element has the height $k+1$ and is called the *top*, while the first k-elements form the *left slope*, and the last k-elements form the *right slope*. Similarly an (a,b)-product of length $2k+2$ (where $k \geqslant 0$) is represented by a trapezium with base angles of 45^0; in this case there are two *top-elements* of height $k+1$, and two slopes of length k each. It may be emphasised that in neither case the top-elements belong to any slope. (a, b)-products of length 1 or 2 have therefore no slopes, whereas every product of this kind has a *top*. If the length is odd, the two elements of height 1 are either both a-elements or both b-elements; if the length is even, one of them is an a-and the other a b-element, also the two top-elements of a product of even length are of different kinds.

The generators α_i of G are supposed to be represented in this manner. Any element $\neq 1$ of G can be represented as an α-product in which no product of consecutive α-factors is equal to 1. In particular, there are no consecutive $\alpha_i \, \alpha_i^{-1}$, and if α_i is of finite order r, there is no sequence of r consecutive factors α_i in the product. To represent an α-product, the triangles and trapezia corresponding to the α-factors are put on the line corresponding to their order in the α-product. If the elements

on the places of contact are of the same kind, they will be amalgamated, and the two ends are pushed together; if in particular the elements are inverse, the corresponding ends are cut off and the two elements coming now in contact are amalgamated, etc. Apply this operation first to products α^m. If length (α) is even, no amalgamation is possible; if length (α) is odd and α is of the type $\sigma a \sigma^{-1}$ (or $\sigma b \sigma^{-1}$), then the $2m-2$ inner slopes disappear, and it remains a triangle $\sigma a^m \sigma^{-1}$ with the top a^m (indeed $a^m \neq 1$, as m is smaller than the order r); if length (α) is odd but the slopes of α are not inverse to each another, the amalgamation in α^m stops even before reaching the top. Hence in a product α^m, only slope elements of the inner slopes can possibly be struck out, the top-elements might be affected but not annihilated, whereas the outer slopes are unchanged. The same holds for products $\delta = (\alpha_2^{\pm 1} \alpha_1^{\pm 1})^{\pm 1}$, since length $\delta \geqslant n_2$; the amalgamation of two consecutive α's (of the same type or of different types) stops latest when one of the tops is reached. If now length α is even, its left top-element cannot be annihilated unless either its α-neighbour to the left, or the right top is annihilated; the corresponding holds for the right top-element. Hence if n_1 and n_2 are even, no top-element can be annihilated, and therefore every α-product must contain even in its reduced form, a's and b's. It is therefore impossible to represent elements a or b by an α-product. Similarly when n_1 is even and n_2 is odd. In the case when α_2 is not of the type $\sigma a \sigma^{-1}$ (or $\sigma b \sigma^{-1}$), the amalgamation stops on the slopes of α_2 and latest on the tops of α_1, but no element comes into contact with an element not belonging to the neighbouring α-factor. If however $\alpha_2 = \sigma a \sigma^{-1}$, then length $(\alpha_1^{\pm 1} \alpha_2^m)^{\pm 1} \leqslant n$; the amalgamation stops therefore on the outer slopes of the triangles α_2^m and latest on the tops of $\alpha_1^{\pm 1}$. Hence no element a or b can be represented by an α-product when n_1 is even, and therefore n_1 must be odd. Suppose $n_1 = 2k+1 > 1$,

then without loss of generality $\alpha_1 = a_1 \ldots a_2$; therefore length $(a_2\alpha_1a_2^{-1}) < n_1$, contrary to (8). Hence $n_1 = 1$. Suppose without loss of generality that $\alpha_1 = a$. If $n_2 = 2m+1 > 1$, then either $\alpha_2 = b_1 \ldots b_2$, or $\alpha_2 = a_3 \ldots a_4$. In the first case no amalgamation takes place between a and α_2, and it is therefore impossible to represent an element b by an α-product. In the second case, length $(a_4\alpha_1a_4^{-1}) = 1$, length $a_4\alpha_2a_4^{-1} < n_2$; thus n_2 is either even or equal to 1. Let n_2 be even, say $\alpha_2 = b \ldots a'$, then $a'a^m \neq 1$, otherwise length $(\alpha_2\, \alpha_1^m) < n_2$; moreover $a'a^ma'^{-1} \neq 1$ for $a^m \neq 1$. In a product in which $\alpha_2^{\pm 1}$ and $\alpha_1^m (\neq 1)$ alternate, the amalgamation stops therefore at height 1, and no element b can be expressed by it. Hence $n_2 = 1$. This completes the proof. The method developed here can also be used in other cases.

3. THEOREM. *Suppose $G = A*B$; let $\{c, d\}$ be the subgroup of G generated by any two elements c and d of G; if $f \neq 1$ is an element of the commutator subgroup $C\{c, d\}$, and f belongs to A, then $\{c, d\}$ is a subgroup of G.*

This theorem has been proved by A. Kurosch for the case, when $f = cdc^{-1}d^{-1} \neq 1$.

PROOF. From a corollary of the * "Untergruppensatz" it follows that $\{c, d\} = U*V$, where $U = A \cap \{c, d\}$. Now U contains $f \neq 1$; therefore $m(U) > 0$, and as $m\{c, d\} = 2$, either $m(U) = m(V) = 1$, or $m(U) = 2, m(V) = 0$. In the latter case, $U = \{c, d\} \subset A$. It suffices therefore to prove that the first case is impossible. Now

$$C(U) = C\{c, d\} \cap U, \tag{9}$$

but U is cyclic and therefore $C(U) = 1$, whereas the right side of (9) contains $f \neq 1$. Hence the theorem.

The theorem can also be proved without the help of

* R. Baer and F. W. Levi, Freie Produkte und ihre Untergruppen, *Compositio Mathematica* 3 (1936), 395.

the formula* (9). Indeed as U and V are cyclic, the factorgroup $F=\{c, d\}/C\{c, d\}$ is isomorphic to the direct product $U \times V$. By this isomorphism, f is mapped on itself, contrary to the supposition that f belongs to $C\{c, d\}$, which implies that f is mapped on 1.

4. The theorem of § 3 is an instance of a statement of the following type: "If the elements $f_1, \ldots$ of $A*B$ belong to A, then $c_1, \ldots$ belong to A. It is known† that if $f \neq 1$ belongs to A, then all the elements of $A*B$ which are commutative with f belong to A. Moreover,

If cd and $dc \neq 1$ belong to A, then c and d belong to A.

Proof. $[c, d]=cdc^{-1}d^{-1}$ belongs to A. If $[c, d] \neq 1$ the statement follows from § 3 (and even from Kurosch's lemma). If $[c, d]=1$, then c and d are commutative and therefore commutative with $dc \neq 1$. Hence d and c belong to A.

The method of § 3 can also be used to investigate the relations which may exist between elements $a \neq 1$ of A and elements c of $A*B$ which do not belong to A. E.g.

If $f(a, c)=1$, then $\{a, c\}$ contains elements of a finite order > 1.

To prove this statement observe that $\{a, c\}=U*V$, where $U=\{a, c\} \cap A$, as also that V contains elements $\neq 1$. U and V are therefore cyclic groups. Suppose that both the cycles are infinite, then $\{a, c\}$ must be a free group with 2 free generators, but a group of this kind cannot‡ be represented by two non-free generators a, c. Hence at least one of the cycles U or V is finite.

* F. W. Levi, The Commutatorgroup of a free group, *Jour. Ind. Math. Soc.* 4 (1940) 136-44.

† *Jour. Ind. Math. Soc.* loc. cit.

‡ *Math. Zeit.* loc. cit.

CONTRIBUTION TO THE RELATIVISTIC PROBLEM OF n BODIES (I)

BY

C. RACINE, *Madras*

[Received 5 July 1941]

It is well known that the relativistic problem of n bodies cannot be stated in a precise manner without assuming a certain number of more or less arbitrary hypotheses in order to obtain the abstraction of material points. Two attempts have been made recently to arrive at the schematic case of n points. The first one is due to Prof. T. Levi Civita* and is based upon the method of approximations used in 1916 by Droste and de Sitter but with rather too much haste as is shown by Levi Civita. The second one is due to Prof. A. Einstein† and is based upon a new method of successive approximations. It leads to results different from those arrived at by Levi Civita, in particular with regard to the secular acceleration of the centre of masses‡.

To study the same problem, I have of late developed a method which I think, is quite new. My results agree with those of Levi Civita. It seems to me that the principal advantage of this method lies in the manner in which the abstraction of material points is obtained. The hypotheses needed are, I think, a little less restrictive than those assumed by Levi Civita.

In this paper I shall explain the principle of the method and treat its application to the first order of

* Levi Civita, *Amer. Jour. Math.* 59 (1937), 9-22.

† Einstein, Infeld and Hoffman, *Annals of Math.* (2) 39 (1938), 64-100.

‡ Levi Civita, *Amer. Jour. Math.* 59 (1937), 225-34. Also, see complement by Robertson, to the article by Einstein etc. in the same volume, p. 101.

approximation. The second approximation and a comparison with the method of Levi Civita will be dealt with in a second paper.

I. GRAVITATIONAL EQUATIONS.

Let us suppose that, in Einstein's space-time, it has been possible to choose the coordinates in such a manner that the linear element is

$$ds^2 = V^2 dt^2 - \sum_{i,j}^{1,2,3} g_{i,j}\, dx^i dx^j, \tag{1}$$

where V and the g_{ij}'s are functions of t, x^1, x^2, x^3, continuous with their first and second derivatives in the whole manifold except on the n varieties limiting those portions of it in which there is matter. On these varieties, only the second derivatives may be discontinuous. We shall call t the "time variable" and the x's the "space variables". The manifolds t=const. will be called "space sections" and their orthogonal trajectories—in the sense of the Geometry of Riemann spaces—will be called the "t-paths".

It will be convenient to denote by R^β_α ($\alpha, \beta = 0, 1, 2, 3$) the components of the contracted Riemannian curvature tensor of the space-time. We shall use the following definition of this tensor:

$$R^\beta_\alpha = \sum_{\lambda=0}^{3} R^{\beta\lambda}_{\cdot\cdot\,\alpha\lambda},$$

with

$$x^0 = t, \quad R^\alpha_{\beta\gamma\delta} = \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} - \frac{\partial \Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} + \sum_\lambda \left(\Gamma^\lambda_{\beta\gamma}\Gamma^\alpha_{\lambda\delta} - \Gamma^\lambda_{\beta\delta}\Gamma^\alpha_{\lambda\gamma} \right).$$

By $\bar{R}^j_i$ ($i, j = 1, 2, 3$) we shall denote the components of the similar tensor relative to the space sections. Further, the Greek indices will run over the numbers 0, 1, 2, 3,—0 being relative to the variable t; the Latin indices running only over the numbers 1, 2, 3. The well-known

summation convention will be used exclusively for summations with respect to Latin indices. For instance $x^i x_i$ stands for $\sum_{i=1}^{3} x^i x_i$. With these conventions we put

$$\Omega_{ij} = \frac{1}{2V}\frac{\partial g_{ij}}{\partial t}, \; \Omega^j_i = g^{jh}\Omega_{hi}, \; K = g^{ih}\Omega_{hi}, \; H^2 = \Omega^h_k \Omega^k_h,$$

$$\Pi^j_i = \Omega^j_i - g^j_i K, \; 2\bar{R} = \sum_1^3 \bar{R}^i_i.$$

If covariant derivation *in the space sections* is denoted by a comma, the semi-colon denoting covariant derivation in space-time, we get readily the following identities:

$$-R^j_i = \bar{R}^j_i + \frac{V^j_{,i}}{V} - K\Omega^j_i - \frac{1}{V}\frac{\partial \Omega^j_i}{\partial t} \; ; \; -R^0_i = \frac{1}{V}\Pi^h_{i,h} \tag{2}$$

$$\sum_1^3 R^i_i - R^0_0 = 2\bar{R} + H^2 - K^2; \; \frac{\partial \Pi^j_i}{\partial t} = \Pi^j_{i;0}.$$

If the components of the energy tensor are denoted by T^β_α, and if k stands for $8\pi f$ where f is the gravitational constant of the classical law of attraction, Einstein's gravitational equations are

$$R^\beta_\alpha = -k(T^\beta_\alpha - g^\beta_\alpha T), \quad 2T = T^0_0 + \sum_1^3 T^i_i.$$

Putting

$$A^j_i = V\bar{R}^j_i + V^j_{,i,}, \; A = g^{ih}A_{ih}, \; B^j_i = A^j_i - g^j_i A, \tag{3}$$

we can write these gravitational equations in the form

$$B^j_i - kV\Pi^j_i - \Pi^j_{i\,;0} = kV(T^j_i + g^j_i T^0_0) = kS^j_i\,;$$

$$\Pi^h_{i,h} = kVT^0_i\,; \quad 2\bar{R} + H^2 - K^2 = -2kT^0_0. \tag{4}$$

All quantities being expressed tensorially, these equations are invariant for the following transformations, which will be called transformations (f):

$$x^i = f_i(y^0, y^1, y^2, y^3), \; (i = 1, 2, 3), \; t = hy^0,$$

h being a constant.

With any system of coordinates other than the first one, the tensor with components Ω_{ij} has to be defined

otherwise. But there is no difficulty in doing so, for we notice at once that the second fundamental form of the space sections with respect to space-time, with the first coordinates, is $\Omega_{hk}dx^h dx^k$ and this expression is invariant for any transformation (f). As to V it can be defined by the linear element of the t-paths $ds^2 = V^2 dt^2$, which is also invariant for the transformations (f).

2. ON CERTAIN SURFACE INTEGRALS.

The very form of the equations (4) suggests that certain surface integrals play an important part in the gravitational theory. In this section we shall refer space-time to the first system of coordinates (t, x^1, x^2, x^3).

Let a vector $\vec{l}\,(0, l^1, l^2, l^3)$ be defined at every point of space-time by the only condition that its components and their first partial derivatives vary continuously. Such a vector field exists, for instance we can put $l^1 = 1$, $l^2 = 0$, $l^3 = 0$.

Further let us determine a second vector field $(0, \xi^1, \xi^2, \xi^3)$ by the conditions

$$\xi_i = g_{ih}\xi^h = \frac{\partial \phi}{\partial x^i}; \quad \xi^h_{,h} = \Delta\phi = l^h_{,k}\Pi^k_h, \tag{5}$$

where Δ stands for the operator

$$g^{ij}\left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma^h_{ij}\frac{\partial}{\partial x^h}\right).$$

Such a fields exists*.

Now we shall consider the surface integral

$$I(l) = \iint_\Sigma (l^i \Pi^h_i - \xi^h)\, d\sigma_h, \tag{6}$$

where $d\sigma_1$ stands for $\sqrt{g}\, dx^2 dx^3$, $d\sigma_2$ and $d\sigma_3$ being obtained from $d\sigma_1$ by a cyclic permutation of the indices; g is the usual determinant whose constituents are the g_{ij}'s; Σ is a

* The proof which is a long one will be published in another paper. Not only the existence of the vector field $(0, \xi^1, \xi^2, \xi^3)$ but also the continuity of the quantities ξ^i will be proved.

regular* closed surface in a space section having no point in common with the regions occupied by masses. Such a surface integral has a meaning if we suppose that the g_{ij}'s are nearly equal to zero if $i \neq j$, and to 1 if $i=j$, and if we choose a "positive" side on the surface. We shall choose here the "exterior" side, as the positive one.

Applying the divergence theorem and taking into account the continuity of the first derivatives of the g_{ij}'s and of the ξ^h's we get†

$$I(l) = \iiint_\Omega l^i \Pi^h_{i,h} \sqrt{g}\, dx^1 dx^2 dx^3 = k \iiint_\Omega l^i T^0_i V \sqrt{g}\, dx^1 dx^2 dx^3 \tag{7}$$

Ω being the volume enclosed by Σ. The integral is therefore equal to zero if Σ does not contain masses in its interior.

Deriving (6) with respect to t and making use of the gravitational equations, we obtain

$$\frac{\partial I}{\partial t} = \iint_\Sigma \left(l^i B^h_i + \frac{\partial l^i}{\partial t} \Pi^h_i - \frac{\partial \xi^h}{\partial t} - KV\xi^h\right) d\sigma_h. \tag{8}$$

This formula will be called the first fundamental formula. I shall show in another paper how it can be applied to the problem of n bodies. In this paper and in the following one I shall make use of another formula which could be obtained from (8) by applying the divergence theorem to the second member, but it is much easier to derive it from (7) and from the relation

$$T^0_{i;0} = -T^h_{i;h}.$$

* A regular surface is a surface defined by the equations

$$x^1 = f_1(u, v),\ x^2 = f_2(u, v),\ x^3 = f_3(u, v),$$

where the functions f_i's are continuous with their first partial derivatives, the Jacobians $\dfrac{D(f_1, f_2)}{D(u, v)}$ etc., never vanishing simultaneously.

† (f) *Cf.* for instance E. Cartan, *Lecons sur la Geometrie des Espaces de Riemann*, Gauthier-Villars, 1928.

From (7) it follows that, if u^0, u^1, u^2, u^3 are the components of the "generalized" velocity

$$\left(u^0 = \frac{dt}{ds},\ u^i = \frac{dx^i}{ds},\ i = 1,\ 2,\ 3\right),$$

$$\frac{\partial I}{\partial t} = k \iiint_\Omega \frac{\partial (l^i T_i^0 V \sqrt{g})}{\partial t}\, dx^1 dx^2 dx^3 + k \iint_\Sigma \frac{u^h}{u^0}\, l^i T_i^0 V d\sigma_h, \tag{9}$$

the surface integral being defined exactly as in (6).

Let us now put

$$T_i^0 = \rho u_i u^0,\ \ dI = \sqrt{g}\, dx^1 dx^2 dx^3,$$

ρ being the density "at rest" of the matter. The last surface integral can be simplified and written in the form

$$\iint_\Sigma l^i T_i^h V d\sigma_h.$$

Then as we have

$$VT^0_{i;0} = V \frac{\partial T_i^0}{\partial t} + T_i^0 \frac{\partial V}{\partial t} + T_i^h V_h - T_0^0 V_i - T_h^0 V^2 \Omega_i^h \tag{10}$$

$$VT^h_{i;h} = VT^h_{i,h} + T_i^0 K V^2 - T_0^h \Omega_{hi};$$

and as by the divergence theorem, with the same definitions as in (6) for the surface integral,

$$\iiint_\Omega (l^i T_i^h V)_{,h} d\tau = \iint_\Sigma l^i V T_i^h d\sigma_h, \tag{11}$$

and

$$l^i_{;0} = \frac{\partial l^i}{\partial t} + l^k V \Omega_k^i, \tag{12}$$

we obtain readily the formula

$$\frac{\partial I}{\partial t} = k \iiint_\Omega l^i T_0^0 V_i d\tau + k \iiint_\Omega (l^i_{;0} - l^k V \Omega_k^i)\, T_i^0 V d\tau + \iiint_\Omega l^h_{,k} T_h^k V d\tau. \tag{13}$$

This will be called the second fundamental formula. It is very easy to apply it to problems of celestial dynamics, and in particular to obtain by means of it a method of arriving at the abstraction of material points. The results derived from it are also easily compared with those

of Levi Civita. But the first fundamental formula seems to me to be capable of applications which, if much more delicate, would probably throw a new light on the relativistic problem of n bodies, leading to the determination of their motion by surface integrals *depending only on the "exterior" field.*

REMARK. If we call "initial conditions" the values of the g_{ij}'s and of the Ω_{ij}'s on the one side, of V on the other side* and finally of the T_α^β's for a given value of t, we can state the

THEOREM. *$\partial I/\partial t$ is a function of the initial conditions.*

3. GENERAL HYPOTHESES ABOUT THE METRIC AND THE VELOCITIES.

In order to apply the Relativity theory to the particular case of the problem of n bodies, a set of hypotheses has been proposed by De Sitter which all the writers treating the question have ever since adopted and which Einstein in his recent memoir† has systematized.

They can be stated as follows: The quantities V and g_{ij}— or more generally the quantities $\overset{*}{g}_{\alpha\beta}$'s† deduced from the first metric by certain transformations (f)—are analytic in $\sqrt{k}$. More precisely, c denoting the velocity of light in vacuum,

$\overset{*}{g}_{00}$, $\dfrac{\overset{*}{g}_{0i}}{\sqrt{k^3}}$, and $\overset{*}{g}_{ij}$ are analytic in k, and for $k=0$

$$\overset{*}{g}_{00}=c,\ \overset{*}{g}_{ij}=0\ (i\neq j),\ g_{ii}=-1.$$

* To determine V we shall have, in general, to state a geometric condition to be satisfied by the space sections considered as subvarieties of the space-time; for instance, we can take these space-sections to be "extremal" varieties of the space-time. The condition that the space-section on which we define the initial conditions be "extremal" is $K=0$. The condition that those in the vicinity be also extremal will suffice to determine V. The treatment of this question is a very long one.

† loc. cit.

Further any derivation with respect to the x's is obtained by term by term derivation. On the contrary a derivation with respect to t is obtained by: (1) term by term derivation, (2) multiplying the series thus obtained by $\sqrt{k}$. For instance, if we put $V=c-kw+\ldots$ then

$$\frac{\partial V}{\partial x^1}=-k\frac{\partial w}{\partial x^1}+\ldots,\quad \frac{\partial V}{\partial t}=k^{\frac{1}{2}}\left[-k\frac{\partial w}{\partial t}+\ldots\right].$$

Finally the space components of the generalized velocity are of the same order of smallness as $k^{\frac{1}{2}}$.

4. APPLICATION OF THE FUNDAMENTAL FORMULA.

This application is made by considering the successive approximations of the two members of that formula, which are analytic in $\sqrt{k}$, and identifying the coefficients of the same powers of $\sqrt{k}$.

I shall consider only the first approximation in this paper. A detailed study of the second approximation is extremely interesting but it would be too long to develop the calculations here. I shall treat it in a second paper. But the method followed in this section will give us sufficient idea of how the fundamental formula leads most naturally to the equations of motion of the problem of n bodies in the Relativity Theory.

Let us put $l^1=1, l^2=l^3=0$, and $T_\alpha^\beta=\rho u_\alpha u^\beta$, ρ and the u^α's having the same meaning as above.

Neglecting terms of the third order in $\sqrt{k}$

$$\xi^h=0,\ V=c-kw,\ T_i^0=-\rho\frac{u^i}{c},\ I(l)=-k\iiint_\Omega \rho\frac{u^1}{c}\,dx^1dx^2dx^3,$$

Ω denoting as above the volume occupied in the space-section considered by a certain celestial body.

In a first approximation, space-time is referred to a "galilean" frame of reference. Therefore the classical notions of centre of forces, sum of a system of vectors, velocity of the centre of gravity, etc. still hold good.

Putting

$$\overline{u^1} = \frac{\iiint_\Omega \rho u^1 dx^1 dx^2 dx^3}{\iiint_\Omega \rho dx^1 dx^2 dx^3} \text{ and } \mu_0 = \iiint_\Omega \rho dx^1 dx^2 dx^3$$

we get

$$I(l) = -\mu_0 \overline{u}^1 / c$$

and $\mu_0 \overline{u}^1/c$ can be interpreted as in classical dynamics.

Hence to the first order of approximation

$$\frac{d}{dt}(\mu_0 \overline{u}^1/c) = \iiint_\Omega \frac{\partial w}{\partial x^1} dx^1 dx^2 dx^3$$

and by a permutation of the Latin indices

$$\frac{d}{dt}(\mu_0 \overline{u}^i/c) = \iiint_\Omega \rho \frac{\partial w}{\partial x^i} dx^1 dx^2 dx^3, \ (i = 1, 2, 3). \quad (14)$$

Finally from the gravitational equations, we get to the first order

$$\Delta w = \sum_i \frac{\partial^2 w}{\partial (x^i)^2} = -\frac{k\rho}{2} = -4\pi f \rho. \quad (15)$$

Now (14) and (15) are precisely the Newtonian laws of motion of celestial bodies. They are deduced, through the fundamental formula (13) from Einstein's equations with a degree of generality which, as far as I know, is new.

To the first order of approximation, the reduction of the celestial bodies to material points is done exactly as in classical celestial dynamics.

CONTRIBUTION TO THE RELATIVISTIC PROBLEM OF n BODIES (II).

BY

C. RACINE, *Madras*

[Received 13 September 1941]

In the previous paper, I have established two "fundamental" formulae, the second of which will now be applied to the relativistic problem of n bodies. Using the same notations as in that paper, but taking units of length and time so that the velocity of light in vacuum is equal to 1, the second fundamental formula can be written as

$$\frac{dI(\vec{l})}{dt} = -k\iint_{\Sigma} l^i V T_i^h \, d\sigma_h + k\iint_{\Sigma} l^i V T_i^0 \frac{u^h}{u^0} d\sigma_h$$

$$+k\iiint_{\Omega} l^i T_0^0 V_i d\tau + k\iiint_{\Omega} [l^i_{;0} - V l^k \Omega^i_k] V t_i^0 d\tau + k\iiint_{\Omega} V l^i_{,k} T_i^k d\tau,$$

where

$$I(\vec{l}) = k\iiint_{\Omega} V l^i T_i^0 d\tau. \qquad (1)$$

Throughout this paper $\vec{l}$ will denote a vector whose contravariant coordinates are the constituents of one of the rows of the matrix

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

1. *Energy tensor.* In the first application of (1), which I have made at the end of my first paper, I put simply

$$T_\alpha^\beta = \rho u_\alpha u^\beta,$$

where the u^{α}'s denote the contravariant components of the "generalized" velocity with respect to a frame of reference such that

$$ds^2 = V^2 dt^2 - \sum_{1}^{3} g_{ij} dx^i dx^j. \qquad (2)$$

Such an expression of the energy tensor is certainly to be avoided in a problem of celestial dynamics for, as is well known, each particle of a celestial body would then move as a free particle. It is not possible to speak of perfectly rigid bodies in the relativity theory. Yet we are certainly allowed to suppose that the bodies are rigid as a first approximation. According to the method of approximation explained in the first paper, this means that if we put in (2): $V=1$, $g_{ii}=1$ $(i=1,2,3)$, $g_{ij}=0$ $(i \neq j)$ and consider therefore the space time as "galilean", the velocity distribution will be exactly that of a rigid body. As a consequence, the orthogonal projections of the generalized velocities on the space sections will be to the first order (in $\sqrt{k}$), a vector field defining a translation, and a rotation about an axis. It is obviously impossible without further hypotheses about the dimensions of the bodies and their mutual distances to assume that the same conclusion holds good to the third order.

Now each portion of a celestial body is acted upon in first and second approximations by gravitational forces determined by volume integrals, and by the "reactions" of the neighbouring portions. These reactions can only be expressed by surface integrals extended over the surface limiting the portion considered. Such surface integrals will obviously be of the form

$$\iint l^h p_h^k d\sigma_k$$

and it is therefore necessary to assume that the energy tensor is the sum of two terms, the first one determined

by the momentum, the other depending on the tensor p_α^β. We shall put

$$T_\alpha^\beta = \rho u_\alpha u^\beta + p_\alpha^\beta.$$

It is not necessary at this stage to determine completely the tensor p_α^β. Its components to the third and fourth orders would be easily determined. It will be sufficient for the purpose of this paper to subject it to certain conditions which are imposed by the nature of the problem.

Since in first approximation the problem of celestial dynamics is a problem of classical dynamics we must have, as in the first paper

$$T_0^0 = 2T = \rho$$

and therefore

$$\sum_1^3 p_i^i = 0, \; p_0^0 = 0,$$

at least to the second order.

We shall therefore put*

$$(a) \qquad \sum_0^3 p_\alpha^\alpha = 0.$$

Further,

(b) we shall suppose that in a frame of reference with respect to which the body Ω is at rest, at points of Ω

$$\bar{p}_0^0 = 0, \; \bar{p}_i^0 = 0, \; (i = 1, 2, 3),$$

$\bar{p}_\beta^\alpha$ denoting the same tensor as previously but referred to the present frame of reference. On account of the hypotheses on the velocities,

(b') the p_i^j's are of order 2 at least, the p_i^0's are of order 3 at least, and p_0^0 is of order 4 at least.

Finally we shall assume that

* *Cf.* H. Weyl, *Math. Zeit.* (1922), 134—45.

(*c*) on the surfaces limiting the celestial bodies, the p^{β}_{α}'s vanish.

As a consequence of (*c*) the surface integrals in (1) vanish, for they are equal to

$$k\iint_{\Sigma} l^i V\left(p^0_i \frac{u^h}{u^0} - p^h_i\right) d\sigma_h.$$

2. *Metric to the second order.* With the hypotheses about the order of the terms of the metric, this will be in the form

$$(1-2U+\ldots)\, dt^2-(1+2U+\ldots).\sum_1^3 (dx^i)^2.$$

From the gravitational equation

$$2\overline{R}+H^2-K^2=-2kT^0_0$$

we get, to the second order

$$\overline{\triangle}\, U=-4\pi f\rho,$$

where $\overline{\triangle}$ stands for $\sum_1^3 \frac{\partial^2}{(\partial x^i)^2}$. Hence

$$U(a^1, a^2, a^3)=\iiint f\rho dx^1dx^2dx^3/r,$$

with

$$r^2=\sum_1^3 (x^i-a^i)^2.$$

Now if the body Ω were at rest with respect to the frame of reference, the "mass at rest" contained in a volume $dx^1dx^2dx^3$ would be

$$d\mu_0=f\rho\sqrt{g}dx^1dx^2dx^3.$$

Since the body is moving, if the velocity of the infinitesimal volume $dx^1dx^2dx^3$ is v, then from the first principles of the Relativity theory, the mass at rest corresponding to that volume will be

$$d\mu_0=f\rho\sqrt{g(1+v^2)}dx^1dx^2dx^3. \quad (3)$$

Consequently if we put

$$u=\iiint d\mu_0/r \tag{4}$$

we have

$$U=u-\iiint (3u+v^2/2)\,d\mu_0/r$$

to the fourth order.

The metric, to the second order, will therefore be assumed to be

$$ds^2=(1-2u)\,dt^2-(1+2u)\sum_{1}^{3}(dx^i)^3. \tag{5}$$

It is much more satisfactory from a systematic point of view to suppose that the mass μ_0 but not the density ρ is given as a function of the domain. That function will evidently be of order 2. Moreover it will satisfy a remarkable relation, for, being the "mass at rest", it follows that if we denote by D a derivation in the direction of the "generalized velocity",

then $$D=\frac{\partial}{\partial t}+\frac{dx^i}{dt}\frac{\partial}{\partial x^i},$$

and $$D\,(d\mu_0)=0. \tag{6}$$

In what follows we shall put

$$\iiint_{\Omega_i} d\mu_0=m_i\,,$$

m_i being the mass at rest of the body Ω_i.

3. *Supplementary hypothesis about energy tensor.*

With the notation just introduced we have

$$I\,(\vec{l})=8\pi\iiint l^i u_i\,d\mu_0+k\iiint V\,l^i p_i^0\,d\tau. \tag{7}$$

Now a fourth hypothesis which seems to be a most natural one may be assumed with the result that the second integral in (7) will vanish to the fifth order.

If we suppose that to the first approximation the body is symmetric with regard to a point and that its angular momentum is zero, then the tensor $\overline{p}^{\beta}_{\alpha}$ will define a reaction opposed to the action of the body upon itself, at least to the first approximation. Thus if P and Q are two symmetric points in the body we must have

$$\overline{p}_{ih}\ (P) = -\overline{p}_{ih}\ (Q)$$

at least to the second order. Now with respect to the first frame of reference we have, to the third order

$$p^0_i = \overline{p}_{ih}\ u^h$$

and therefore, the angular velocity being zero

$$p^0_i\ (P) = -p^0_i\ (Q)$$

to the third order.

If the angular momentum cannot be neglected in the first approximation, then we shall put again, (this being precisely the fourth hypothesis about the energy tensor)

$$(d) \qquad \overline{p}_{ih}\ (P).\ u^h(P) = -\overline{p}_{ih}(Q).\ u^h(Q).$$

As a consequence, the vector $\vec{l}$ being a fixed vector we have, to the third order

$$\iiint l^i\ p^0_i\ dx^1dx^2dx^3 = 0$$

and therefore, to the fifth order

$$I\ (\vec{l}) = 8\pi \iiint l^i u_i\ d\mu_0. \qquad (8)$$

In the first paper I have defined $I\ (\vec{l})$ by means of a surface integral extended over a closed regular surface of the "exterior" space time. Putting

$$m^i l^h\ \overline{u}_h = \iiint_{\Omega_i} l^h u_h\ d\mu_0 = l^h \overline{p}_h$$

and interpreting $\overline{p}_h$ as the "momentum" of the body Ω_i in the direction of the vector $\vec{l}$, we have

$$I\ (\vec{l}) = 8\pi l^h\ \overline{p}_h, \qquad (8)$$

and therefore we have the

THEOREM.—*The momentum of a celestial body is entirely determined to the fifth order by the exterior field.*

If we define $\frac{d(l^h \overline{p^h})}{dt}$ as the "gravitational force" acting upon the body we have also the

THEOREM.—*The gravitational force acting upon a celestial body is, to the sixth order, entirely determined by the exterior field.*

This is an immediate consequence of the "first" fundamental formula given in my first paper.

4. *Differential equations of the motion.* The second member of (1) can now be written in the following manner: the first two surface integrals vanish as has been shown; the first volume integral is obviously

$$8\pi \iiint l^i \sqrt{1+v^2}\, V_i d\mu_0. \tag{9}$$

It gives the most important contribution to the motion.

To the sixth order, the second volume integral is

$$8\pi \iiint l^h \,[\Gamma^i_{h0} - \Omega^i_h]\, u_i\, d\mu_0 = 8\pi \iiint l^h \frac{\partial g_{0i}}{\partial x^h} \frac{dx^i}{dt}\, d\mu_0. \tag{10}$$

Finally the third one, introducing the elements of the metric of order 2 becomes

$$-8\pi \iiint l^h \frac{\partial u}{\partial x_h} v^2\, d\mu_0, \quad v^2 = u^i u_i \tag{11}$$

to the sixth order.

Now

$$l^i u_i = \sum_0^3 l_\alpha u^\alpha = l_0 \frac{dt}{ds} + g_{ih}\, l^h \frac{dx^i}{ds},$$

and therefore to the third order

$$l^i u_i = l^i \left[g_{0i} - (1+2u) \frac{dx^i}{dt} \right] \frac{dt}{ds}. \tag{12}$$

As

$$V^2\left(\frac{dt}{ds}\right)^2 = 1+v^2+\dots,$$

we have

$$\frac{dt}{ds} = 1+u+\frac{v^2}{2}+\dots.$$

Hence to the fifth order

$$I(\vec{l}) = 8\pi\iiint l^i\left[g_{0i}-\left(1+3u+\frac{v^2}{2}\right)\frac{dx^i}{dt}\right]d\mu_0, \qquad (13)$$

and the fundamental formula yields, to the sixth order

$$\frac{d^2}{dt^2}\iiint l^i x^i\, d\mu_0 = -\iiint_\Omega l^i V_i\, d\mu_0 + P + R, \qquad (14)$$

where,

$$P = -\iiint 3u\, l^i\left[\frac{d^2x^i}{dt^2}\right]_2 d\mu_0 + \iiint l^i\frac{\partial g_{0i}}{\partial t}\, d\mu_0$$

$$R = \iiint l^h\left\{3\frac{\partial u}{\partial x^h} - \left[\frac{d^2x^h}{dt^2}\right]_2\right\}\frac{v^2}{2}\, d\mu_0$$

$$+\iiint l^h\sum_{i=1}^{3}\left(\frac{\partial g_{0h}}{\partial x^i}-\frac{\partial g_{0i}}{\partial x^h}\right)\left[\frac{dx^i}{dt}\right]_1 d\mu_0$$

$$-\iiint l^h\left\{3\, Du + \sum_{k=1}^{3}\left[\frac{d^2x^k}{dt^2}\right]_2\left[\frac{dx^k}{dt}\right]_1\right\}\left[\frac{dx^h}{dt}\right]_1 d\mu_0, \qquad (15)$$

the symbol $[\;]_p$ denoting the approximation to the pth order of the quantity between brackets.

In particular, if, to the first order, the angular momentum of the body considered vanishes, then $[dx^i/dt]_1$ will be constant at time t throughout the body and we shall have

$$m\left[\frac{d^2x^i}{dt^2}\right]_2 = \iiint l^i\frac{\partial u}{\partial x^i}\, d\mu_0.$$

We have not yet obtained a set of differential equations for the motion of the n bodies. This will not be possible without introducing some new hypotheses,

for, in general, as Levi Civita has remarked, the relativistic dynamics is essentially a dynamics of fluids. Thus it would be a difficult problem to determine the shape of a given body at time t to the fourth, even to the second order, knowing its shape at the initial time. Yet to calculate to the sixth order the integral

$$\iiint l^i \frac{\partial u}{\partial x^i} d\mu_0,$$

this is necessary.

However, equations (14) and (15) manifestly suggest a condition under which the problem of the motion of n bodies, to the approximation considered, reduces to the problem of the motion of their centres of mass. These points being denoted by C_i and their coordinates determined by the integrals

$$\frac{1}{m_i}\iiint_{\Omega_i} x^h d\mu_0, \quad (h = 1, 2, 3;\ i = 1, 2, \ldots, n),$$

this condition may be expressed as follows:

Each volume integral of the right hand member of (15) is merely a function of the coordinates of the centre of mass of the corresponding body, and of t.

In particular, if C_i is the centre of mass of the body, this condition will be verified *if for each volume integral the following equation, where $F(P)$ represents the coefficient of $d\mu_0$, holds good to the sixth order*

$$\iiint_{\Omega_i} F(P)\, d\mu_0 = F(C_i) \iiint_{\Omega_i} d\mu_0 = m_i F(C_i). \qquad (16)$$

More generally, let us suppose that the coordinates of C_i are c^1, c^2, c^3, and those of P, x^1, x^2, x^3. Applying Taylor's formula,

$$F(P) = F(C_i) + \sum (x^h - c^h) \frac{\partial F(C_i)}{\partial x^h} + \sum \frac{(x^h - c^h)(x^k - c^k)}{2} \frac{\partial^2 F(C_i)}{\partial x^h \partial x^k} + \ldots \qquad (17)$$

If δ denotes the lower bound of the mutual distances between the C_i's, and λ the upper bound of the diameters of the bodies, we have

$$F(P) = O(\delta^{-1})$$

and the derivatives of F of order $m-1$ are $O(\delta^{-m})$. Therefore if

$$\lambda^n \delta^{-n-1} = O(k^{3/2})$$

we can substitute for $F(P)$ the first n terms of its Taylor's formula neglecting the remainder. Then we get the

THEOREM. *If the integrals*

$$\iiint (x^h - c^h)(x^k - c^k)\, d\mu_0, \ldots \tag{18}$$

are independent of t and behave therefore as constants, and if

$$\lambda^n \delta^{-n-1} = O(k^{3/2}),$$

the motion of the points C_i is determined by differential equations.

In that case we can say that we have reached *the abstraction of material points.**

The conditions stated by Levi Civita in the papers referred to in my previous paper and also in his contribution to the Jubilee Volume of Prof. M. Brillouin† are the following:

A. 1. Hypotheses on the order of the terms. They are the same as in these two papers.

A. 2. The quantity which I have denoted by $\lambda^2\delta^{-2}$ is "negligible."

A. 3. The motion of each body is "roughly a pure translation." Namely, if $\vec{r}$ is the velocity at the centre and $\vec{r} + \triangle\vec{r}$ the velocity at another point of the body, then $\frac{|\triangle r|}{|r|}$ is "negligible."

* If, more generally, the integrals (18) were functions of t, the same result could be stated. But as it is most likely that these functions are extremely difficult to obtain, if they can be obtained at all, I do not consider this case.

† Gauthier-Villars, 1935, pp. 203-12, specially pp. 206-8.

A. 4. "The Newtonian attraction of a body on its centre of mass amounts at most to some one hundredth of the attraction exerted on the same point by all the other bodies of the system."

In his article in the Jubilee Volume of Prof. M. Brillouin, this hypothesis is proposed in a different manner. But in any case the consequence is the same, namely, that the two "self-contributions"

$$\omega(a^1, a^2, a^3) = \iiint_{\Omega_i} d\mu_0/r$$

and

$$\eta(a^1, a^2, a^3) = \iiint_{\Omega_i} d\mu_0 \iiint_{\Omega_i} d\mu_0/r,$$

where a^1, a^2, a^3, being the coordinates of a point interior to Ω_i, can be treated as constants.

It can be easily verified that these hypotheses lead to the conditions (16), and the differential equations deduced from (14) and (15) are those of n geodesics.*

Instead of these hypotheses let us assume the following:

B. 1. The same as A. 1

B. 2. $\lambda^2\delta^{-3}$ is of order >2.

B. 3. $\dfrac{|\triangle r|}{|r|}$ is of order >1.

B. 4. To the first order the function μ_0 is in each body symmetric with respect to the centre of mass.

Then it is readily seen that the equations (16) hold good as in the case of the A-hypotheses and therefore the centres of mass move on geodesics in space-time. Taking into account the self-contributions, the equations of motion are exactly the same as those of Levi Civita.

* *Cf.* J. Chazy, *La theorie de la Relativite et la Mecanique Celeste*, Gauthier-Villars, 1930, tome II, p. 168.

It is important to remark that assuming the more general hypotheses under which the functions $F(P)$ could be replaced by a certain number of initial terms of their Taylor's formulae, *the problem of n bodies still depends on differential equations, but the trajectories of the centres of mass are no longer geodesics of space-time.* They depend on a number of "physical constants" defined analytically by the integrals (18).

5. *Approximation of the metric to the fourth order.* The g_{ij}'s have only to be determined to the second order, as can be seen immediately from (14). Therefore we shall put

$$g_{ij} = 0, \ (i \neq j) \text{ and } g_{ii} = 1 + 2u, \ (i = 1, 2, 3)$$

with

$$u = \iiint \frac{d\mu_0}{r}.$$

To obtain the terms of the fourth order of V, and those of the third order of the g_{0i}'s, we might adopt the ds^2 of de Sitter and this is what Levi Civita has done. Yet in this metric V does not reduce to 1 at infinity and this is already objectionable. Further the space sections are not defined by any intrinsic property.

I have made use of a slightly different metric in which V reduces to 1 at infinity and *the space sections are extremal in space-time.* This metric is determined by

$$V = 1 - u + \frac{u^2}{2} + \iiint \left(u - \frac{3}{2} v^2\right) \frac{d\mu_0}{r}, \quad g_{0i} = \lambda_i + \frac{\partial \theta}{\partial x^i}, \tag{19}$$

where

$$\lambda_i = 4 \iiint \frac{u^i \, d\mu_0}{r}, \quad \theta = \frac{\partial Q}{\partial t}, \quad Q = \iiint r \, d\mu_0.$$

Then

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial g_{0i}}{\partial x^j} + \frac{\partial g_{0j}}{\partial x^i}\right) + g_i^j \frac{\partial u}{\partial t}, \quad \Omega^h_{i,h} = \frac{1}{2} \triangle \lambda_i,$$

and $K = 0$, this condition expressing that the manifolds $t =$ const. are extremal.

It is easily verified that this metric satisfies the gravitational equations to the fourth order.*

We may notice also that with the metric (19), or with de Sitter's metric, the equations of motion defined by the second fundamental formula are to the sixth order entirely independent of the components of the tensor p^{β}_{α}. This is quite remarkable.

I shall conclude this paper by referring to an interesting problem which I have not treated, although I believe it is an important one. Supposing that owing to a certain set of hypotheses the motion of n bodies is determined by differential equations, then the different "forces" acting upon the centres of mass in the sense defined above are given by volume integrals. Some of them may be *partly*† cancelled by a change of coordinates of the (f) class, namely

$$y^i = x^i + X^i\ (x^1, x^2, x^3, t)\ (i = 1, 2, 3), y^0 = t, \qquad (20)$$

the functions X^i $(i = 1, 2, 3)$ being at least $O(k)$ and tending to zero with $1/r$, where $r^2 = \sum_1^3 (x^i)^2$.

Such in particular are the forces corresponding to the integrals

$$\iiint l^i D g_{0i} d_{\mu 0}, \iiint l^h\ [\Gamma^i_{h0} - \Omega^i_h]\ u_i d_{\mu 0}, \qquad (21)$$

when the coordinates are such that the ds^2 is given by (2).

It follows that, to the sixth order, it is impossible to speak of the "gravitational" forces as in the case of the fourth order, or as in classical dynamics. In classical dynamics, such forces are determined by referring the

* The existence of de Sitter's metric is still an open question. In fact it is obtained by a process of successive approximations, and as long as it has not been proved that these approximations are convergent, the validity of the metric may be doubted.

†I mean to say that volume integrals like (21) may be cancelled by the change of coordinates (20). But then the other volume integrals are slightly modified and therefore we cannot say that the corresponding force has been *completely* cancelled.

motion to Galilean axes; but such frames of reference do not exist in Relativity Theory and therefore each frame of reference will necessarily introduce some special set of forces, some of which most likely do not affect the qualitative analysis of the trajectories. Also there exist probably frames of reference which reduce the action of such forces to a minimum. To distinguish between forces which could be called properly "gravitational" because they will be found in any frame of reference, and those which an analogy with classical dynamics would suggest being called "Coriolis" forces; then of determining to what extent the latter might be neglected in a qualitative analysis of the trajectories, is a problem which, as far as I know, has never been studied, nor even perhaps set precisely.*

*In the case of two bodies of masses M and $m, m/M$ being $O(\sqrt{k})$, and the angular momenta being negligible, the force defined by the second integral of (21) vanishes to the sixth order and the perturbation produced by the force corresponding to the first integral is the same as would be caused by a change $\triangle m$ of the mass m, $\triangle m$ being $O(k^2)$. If m is altogether negligible and if the body of mass M has an angular momentum, then secular accelerations are introduced by the second integral of (21). De Sitter has studied the case and shown that these accelerations are so small that they cannot be observed. *Cf.* J. Chazy, *loc. cit.* tome II, 171-5.

129113

A THEOREM ON INTEGRAL FUNCTIONS OF INTEGRAL ORDER (II).

BY

S. M. SHAH, *Aligarh*

[Received 30 June 1941, in revised form 19 November 1941].

1. Let $F(z)$ be an integral function of finite order ρ. We write $F(z) = z^k e^{Q(z)} f(z)$, where $Q(z)$ is a polynomial of degree $q \leqslant \rho$, k a positive integer or zero and

$$f(z) = \prod_1^\infty \left\{ \left(1 - \frac{z}{a_n}\right) \exp\left(\frac{z}{a_n} + \ldots + \frac{1}{p}\left(\frac{z}{a_n}\right)^p\right)\right\}$$

is the canonical product of order ρ_1, and genus p. Let $M(r, F) = \underset{|z|=r}{\text{Max}} |F(z)|$ and $n(r, F-a) = n(r, a)$ the number of zeros of $F(z)-a$ in and on $|z| = r$.

In a previous paper* I proved

THEOREM I. *If $f(z)$ be a canonical product of order and genus $p \geqslant 1$ then*

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{n(r,f)\phi(r)} = 0, \tag{1}$$

where $\phi(x)$ is any positive continuous non-decreasing function of the real variable x, such that

$$\int_a^\infty \frac{dx}{x\phi(x)} \tag{2}$$

is convergent.

In the same paper this result has been extended to integral functions $F(z)$ of integral order ρ for which $\Sigma 1/r_n^\rho$ is divergent. In this note I prove a similar result for

* S M. Shah. *Jour. London Math. Soc.* 15 (1940), 23-31. A correction is necessary on p. 25 in Lemma 2. Replace in line 7 lim $\psi(x)/x$ by lim sup $\psi(x)/x$. The proof has to be modified slightly at the end.

JL

Bm 21, N34

canonical products of order ρ and genus $\rho-1$, and discuss whether the result can be extended to integral functions which are not canonical products. The main result is the following

THEOREM 2. *If $f(z)$ be a canonical product of order ρ and genus $p=\rho-1$ then*

$$\liminf_{r\to\infty}\frac{\log M(r,f)}{n(r,f)\phi(r)}=0, \tag{3}$$

where $\phi(x)$ has the properties stated in Theorem 1.

2. LEMMA 1. *Given any positive continuous non-decreasing function $\phi(x)$ such that*

$$\int_a^\infty \frac{dx}{x\phi(x)}$$

is convergent, there exists a positive continuous non-decreasing function $\psi(x)$ satisfying for $x\geqslant R=$ a constant, (i) $\psi(x)\leqslant\phi(x)$ (ii) $\psi(x)/x^\alpha$ is non-increasing, α being a positive number less than unity (iii) $I=\int_R^\infty \frac{dx}{x\psi(x)}$ is convergent.

PROOF. Let $R\geqslant \mathrm{Max}(1, a)$ be such that $\phi(R)>0$. Let $\psi(x)=\frac{\phi(R)x^\alpha}{2R^\alpha}$ for $R\leqslant x\leqslant R_1$, where R_1 is the first meet of $Y=\phi(x)$ and $Y=\frac{\phi(R)x^\alpha}{2R^\alpha}$ in the interval $x>R$. If they do not meet, we take $\psi(x)=\frac{\phi(R)x^\alpha}{2R^\alpha}$ throughout $x\geqslant R$. $\psi(x)=\phi(R_1)$ for $R_1\leqslant x\leqslant 1+\xi_1$, where $\xi_1(\geqslant R_1)$ is the greatest number such that $\phi(\xi_1)=\phi(R_1)$. We have $\psi(1+\xi_1)=\psi(R_1)=\phi(R_1)$. Further let

$$\psi(x)=\frac{\phi(R_1)x^\alpha}{(1+\xi_1)^\alpha} \text{ for } 1+\xi_1\leqslant x\leqslant R_2,$$

where R_2 is the first meet of $Y=\phi(x)$ and $Y=\frac{\phi(R_1)x^\alpha}{(1+\xi_1)^\alpha}$ in the interval $x>1+\xi_1$,

$$\psi(x)=\phi(R_2) \text{ for } R_2\leqslant x\leqslant 1+\xi_2,$$

where $\xi_2(\geqslant R_2)$ is the greatest number such that $\phi(\xi_2)=\phi(R_2)$. Repeating this process we have

$$\psi(x)=\phi(R_m) \text{ for } R_m\leqslant x\leqslant 1+\xi_m$$
$$=\frac{\phi(R_m)\,x^\alpha}{(1+\xi_m)^\alpha} \text{ for } 1+\xi_m\leqslant x\leqslant R_{m+1},$$

where $\xi_m(\geqslant R_m)$ is the greatest number such that $\phi(\xi_m)=\phi(R_m)$. R_{m+1} is the first meet of $Y=\phi(x)$ and $Y=\dfrac{\phi(R_m)x^\alpha}{(1+\xi_m)^\alpha}$ in the interval $x>1+\xi_m$.

We have for $x\geqslant R$, $\psi(x)>0$ non-decreasing, $\psi(x)\leqslant\phi(x)$ and $\psi(x)/x^\alpha$ is non-increasing. We prove now the convergence of

$$\int_{R_1}^{\infty}\frac{dx}{x\psi(x)}=\sum_1^\infty\int_{R_m}^{R_{m+1}}\frac{dx}{x\psi(x)}$$
$$=\sum_1^{\infty}{}'\left\{\int_{R_m}^{S_m}+\int_{S_m}^{R_{m+1}}\right\}+\sum_1^{\infty}{}''\int_{R_m}^{R_{m+1}}=\sum_1{}'+\sum_2{}'+\sum_3{}''$$

where in $\sum_1'$, $\sum_2'$ summation is to be taken over those values of m for which $S_m=R_m^{1+1/\alpha}<R_{m+1}$ and in $\sum_3''$ summation for those m for which $S_m\geqslant R_{m+1}$. In $\sum_1'$

$$\int_{R_m}^{S_m}\frac{dx}{x\psi(x)}\leqslant\int_{R_m}^{S_m}\frac{dx}{x\phi(x^c)},$$

where $c=\alpha/(\alpha+1)$, for $\psi(x)\geqslant\psi(R_m)=\phi(R_m)\geqslant\phi(x^c)$ since $x<S_m$ and therefore $x^c\leqslant R_m$. Putting $x^c=y$ we have

$$\sum_1{}'\leqslant\sum\int_{R_m}^{S_m}\frac{dx}{x\phi(x^c)}=\sum\frac{1}{c}\int_{R_m^c}^{S_m^c}\frac{dy}{y\phi(y)}<\infty.$$

Similarly

$$\sum_3{}''=\sum\int_{R_m}^{R_{m+1}}\frac{dx}{x\psi(x)}.$$

Here $R_m\leqslant x<R_{m+1}\leqslant S_m$. Hence $\psi(x)\geqslant\psi(R_m)=\phi(R_m)\geqslant\phi(x^c)$.

Hence

$$\sum_3'' < \sum_1^{\infty} \int_{R_m}^{R_m+1} \frac{dx}{x\phi(x^c)} < \infty.$$

Further

$$\sum_2' = \sum_{21} + \sum_{22},$$

where in $\sum_{21}$ the summation is for those m for which $1+\xi_m < S_m$ and in $\sum_{22}$ for those m, if any, for which $1+\xi_m > S_m$.

$$\sum_{21} = \sum \int_{S_m}^{R_m+1} \frac{dx}{x\psi(x)} = \sum \int_{S_m}^{R_m+1} \frac{(1+\xi_m)^\alpha}{\phi(R_m)} \frac{dx}{x^{1+\alpha}}$$

$$< \frac{1}{\alpha} \sum \frac{(1+\xi_m)^\alpha}{\phi(R_m)} \frac{1}{R_m^{1+\alpha}} = \sum_{211} + \sum_{212},$$

where in $\sum_{211}$, summation is for those m for which $\xi_m \leqslant 2R_m$ and in $\sum_{212}$, $\xi_m > 2R_m$. Let this latter set of values of m be denoted by $e(m)$, then

$$\infty > \sum_{e(m)} \int_{R_m}^{\xi_m} \frac{dx}{x\phi(x)} = \sum_{e(m)} \frac{1}{\phi(R_m)} \log \frac{\xi_m}{R_m} > \sum_{e(m)} \frac{\log 2}{\phi(R_m)}.$$

Now

$$(1+\xi_m)^\alpha < 3^\alpha R_m^\alpha \text{ for } \Sigma_{211} \text{ and } (1+\xi_m)^\alpha \leqslant R_m^{1+\alpha} \text{ for } \Sigma_{212}.$$

Hence

$$\sum_{21} < \frac{3^\alpha}{\alpha} \sum_1^{\infty} \frac{R_m^\alpha}{\phi(R_m)\, R_m^{1+\alpha}} + \frac{1}{\alpha} \sum_{e(m)} \frac{1}{\phi(R_m)}$$

$$< \frac{3^\alpha}{\alpha} \sum_1^{\infty} \frac{1}{m\phi(m)} + \frac{1}{\alpha} \sum_{e(m)} \frac{1}{\phi(R_m)} < \infty.$$

Now

$$\sum_{22} = \sum \left\{ \int_{S_m}^{\xi_m} + \int_{\xi_m}^{1+\xi_m} + \int_{1+\xi_m}^{R_m+1} \right\} + \sum \left\{ \int_{S_m}^{1+\xi_m} + \int_{1+\xi_m}^{R_{1+m}} \right\}$$

$$= \Sigma_{221} + \Sigma_{222} + \Sigma_{223} + \Sigma_{224} + \Sigma_{225},$$

where in Σ_{221} to Σ_{223}, summation is for those m for which $S_m \leqslant \xi_m$ and in Σ_{224}, Σ_{225} we have $\xi_m < S_m < 1 + \xi_m$.

Now

$$\sum_{221} = \sum \int_{S_m}^{\xi_m} \frac{dx}{x\psi(x)} = \sum \int_{S_m}^{\xi_m} \frac{dx}{x\phi(x)} < \infty$$

$$\sum_{222} = \sum \int_{\xi_m}^{1+\xi_m} \frac{dx}{x\psi(x)} = \sum \int_{\xi_m}^{1+\xi_m} \frac{dx}{x\phi(R_m)}$$

$$= \sum \frac{1}{\phi(R_m)} \log \left(1 + \frac{1}{\xi_m}\right) \leqslant \sum \frac{1}{\xi_m \phi(\xi_m)} \leqslant \sum \frac{1}{m\phi(m)} < \infty$$

$$\sum_{223} = \sum \int_{1+\xi_m}^{R_{1+m}} \frac{(1+\xi_m)^\alpha}{x^{1+\alpha}\phi(R_m)} < \sum \frac{(1+\xi_m)^\alpha}{\alpha\phi(R_m)} \frac{1}{(1+\xi_m)^\alpha}$$

$$< K \sum_{E(m)} \frac{1}{\phi(R_m)},$$

where $E(m)$ is a set of values of m such that

$$\infty > \sum_{E(m)} \int_{R_m}^{\xi_m} \frac{dx}{x\phi(x)} = \sum_{E(m)} \frac{1}{\phi(R_m)} \log \frac{\xi_m}{R_m}.$$

and since $\xi_m \geqslant S_m > 2R_m$ for all large m, we get $\Sigma_{223} < \infty$, and as in Σ_{222} and Σ_{223} we have $\Sigma_{224} < \infty$; $\Sigma_{225} < \infty$.

Hence

$$\sum_2' = \sum_{21} + \sum_{22} < \infty$$

and so the integral I is convergent.

3. Lemma 2. *If* $0 < \beta < 1$, *then for all* $r \geqslant r_0(A, \beta)$

$$\mathcal{J} = \int_\Delta^r \frac{dx}{x^\beta \phi(x)} < \frac{Ar^{1-\beta}}{\log r},$$

where $\phi(x)$ *is any positive non-decreasing continuous function satisfying* (2).

Proof. From the convergence of the integral, we have

$$\log x < \phi(x) \text{ for all large } x.$$

Hence for $r \geqslant r_0$

$$\mathcal{J} = \int_{\Delta}^{\sqrt{r}} + \int_{\sqrt{r}}^{r} \leqslant \frac{1}{\phi(\Delta)} \frac{r^{\frac{1}{2}(1-\beta)}}{(1-\beta)} + \frac{2}{(1-\beta)} \frac{r^{1-\beta}}{\log r}$$
$$< \frac{A\, r^{1-\beta}}{\log r}.$$

LEMMA 3. *Suppose that the real functions* $\Psi(x)$ *and* $\theta(x)$ *satisfy the following conditions:*

(i) $\Psi(x)$ *is continuous in* (δ, ∞) *where* $\delta > 0$ *except for isolated points where* $\Psi(x)$ *has ordinary left-hand discontinuities,*

(ii) $\Psi(x)$ *is non-increasing as* $x \geqslant \delta$ *increases in any interval between two consecutive discontinuities,*

(iii) $\theta(x)$ *is a positive continuous increasing function for* $x \geqslant \delta$,

(iv) $$\limsup_{x\to\infty} \Psi(x) = \infty; \ \limsup_{x\to\infty} \frac{\Psi(x)}{\theta(x)} = 0.$$

Then we can find a sequence (x_n) *of values of* x *tending to* ∞ *such that the two inequalities*

$$\Psi(x) \leqslant \Psi(x_n), \ (x_1 \leqslant x < x_n)$$
$$\frac{\Psi(x)}{\theta(x)} \leqslant \frac{\Psi(x_n)}{\theta(x_n)}, \ (x > x_n)$$

are satisfied simultaneously.

The (x_n) *are points of discontinuity so that* $\Psi(x_n) = \Psi(x_n+0)$ *and* x_1 *is the first point of discontinuity in* (δ, ∞).

The proof is similar to that of Lemma 2 of my previous paper.

4. We prove

$$\liminf_{r\to\infty} \frac{\log M(r)}{n(r)\psi(r)} = 0, \tag{4}$$

where $\psi(x)$ has been defined in Lemma 1. Theorem 2 would follow since $\psi(r) \leqslant \phi(r)$.

Suppose first that

$$\limsup_{r\to\infty}\frac{n(r,f)\psi(r)}{r^{p+1}}>0. \tag{5}$$

We have then

$$r^{p+1}<An(r,f)\psi(r) \tag{5.1}$$

for infinitely many $r=R_n$ tending to infinity and so*

$$\frac{\log M(R_n)}{n(R_n)\psi(R_n)}<\frac{A\log M(R_n)}{R_n^{1+p}}$$

which tends to zero with $1/n$. Hence

$$\liminf_{r\to\infty}\frac{\log M(r)}{n(r)\psi(r)}=0.$$

Suppose secondly that

$$\lim_{r\to\infty}\frac{n(r,f)\psi(r)}{r^{1+p}}=0. \tag{5.2}$$

We apply Lemma 3. Let β be a constant such that $0<\beta<1-\alpha$; $\theta(x)=x^{\beta}$; $\delta=1$.

$$\Psi(x)=\frac{n(x)\psi(x)}{x^{p+1-\beta}}=n(x)\frac{\psi(x)}{x^{\alpha}}\ \frac{1}{x^{p+1-\alpha-\beta}}.$$

The conditions of Lemma 3 are satisfied, and so puting $x_n=R$ we obtain

$$\frac{n(x)\psi(x)}{x^{p+1-\beta}}\leqslant\frac{n(R)\psi(R)}{R^{p+1-\beta}}\ \text{for } 1\leqslant x_1\leqslant x\leqslant R$$

$$\frac{n(x)\psi(x)}{x^{p+1}}\leqslant\frac{n(R)\psi(R)}{R^{p+1}}\ \text{for } x>R.$$

Hence for $R>x_1$,

$$\begin{aligned}\log M(R,f)<AI(R,f)&=A\int_0^\infty\frac{n(x,f)}{x^{p+1}}\ \frac{R^{p+1}}{(x+R)}dx\\ &\leqslant A\left\{A_1R^p\int_{x_1}^R\frac{n(x)}{x^{p+1}}\,dx+R^{p+1}\int_R^\infty\frac{n(x)}{x^{p+2}}\,dx\right\}\\ &\leqslant A\left\{A_1R^p\frac{n(R)\psi(R)}{R^{p+1-\beta}}\int_{x_1}^R\frac{dx}{x^{\beta}\psi(x)}+n(R)\psi(R)\int^\infty\frac{dx}{x\psi(x)}\right\}\\ &\leqslant A\left\{A_2\frac{n(R)\psi(R)}{\log R}+o\Big(n(R)\psi(R)\Big)\right\}\end{aligned}$$

* A denotes a positive finite constant, not necessarily the same at each occurrence.

by Lemma 2. Hence

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{n(r,f)\psi(r)} \leqslant A \liminf_{r\to\infty} \frac{I(r,f)}{n(r,f)\psi(r)} = 0 \qquad (6)$$

and this completes the proof of (4).

Corollary. *If* $F(z)=z^k e^{Q(z)} f(z)$ *be of integral order* ρ *and genus* $g=\rho-1$, *then*

$$\liminf_{r\to\infty} \frac{\log M(r,F)}{n(r,F)\phi(r)} = 0. \qquad (7)$$

We have $g=\rho-1=\text{Max}(p,\ q)$. It is easily seen that $p=\rho-1$, $q\leqslant\rho-1$ and

$$\log M(r,F) < A\{r^{\rho-1}+\log r\}+\log M(r,f).$$

If now

$$\limsup_{r\to\infty} \frac{n(r,f)\psi(r)}{r^{p+1}} > 0,$$

then $\dfrac{\log M(R_n,F)}{n(R_n,F)\psi(R_n)} < \dfrac{A\{R_n^{\rho-1}+\log R_n\}}{n(R_n,F)\psi(R_n)} + \dfrac{\log M(R_n,f)}{n(R_n,f)\psi(R_n)}$,

where R_n is defined in (5.1).

Hence

$$\liminf_{r\to\infty} \frac{\log M(r,F)}{n(r,F)\ \psi(r)} = 0. \qquad (8)$$

If now

$$\lim_{r\to\infty} \frac{n(r,f)\psi(r)}{r^{p+1}} = 0,$$

then for all large r

$$\log M(r,F) < A\{r^{\rho-1}+\log r\}+AI(r,f) < A_3 I(r,f)$$

and hence from (6)

$$\liminf_{r\to\infty} \frac{\log M(r,F)}{n(r,F)\psi(r)} = 0.$$

Since $\psi(x)\leqslant\phi(x)$, (7) is proved.

5. There are five possibilities when $F(z)$ is of integral order ρ.*

(i) $p\leqslant\rho_1<\rho$; $q=\rho$ (ii) $\rho_1=p=\rho=q$ (iii) $\rho_1=p=\rho>q$
(iv) $\rho_1=q=\rho$, $p=\rho-1$ (v) $\rho_1=\rho$, $q<\rho$, $p=\rho-1$.

* Valiron, *Lectures on the general theory of integral functions*, p. 59.

Combining the above results we have in cases (ii), (iii) and (v)

$$\liminf_{r\to\infty} \frac{\log M(r, F)}{n(r,F)\phi(r)} = 0. \tag{9}$$

In cases (i) and (iv), (9) does not hold.* For functions of fractional and zero order, (9) holds.* It also holds for functions of maximum or minimum type. *It holds for every function $F(z)$ of which the genus is equal to the genus of its canonical product.*

6. It is known that if $F(z)$ be of integral order ρ and a be any number, then†

$$\liminf_{r\to\infty} \frac{\log M(r, F)}{n(r, F-a)} < \infty$$

for every a save possibly a single exceptional value of a. Since $F(z)$ and $F(z)-a$ belong to the same type, we deduce that *if $F(z)$ be of maximum or minimum type, order ρ, where ρ is an integer, then*

$$\liminf_{r\to\infty} \frac{\log M(r, F)}{n(r, F-a)\phi(r)} = 0 \tag{10}$$

for every a. If $F(z)$ be of mean type, then (10) may not hold for one exceptional value of a.

7. The condition (2) on $\phi(x)$ is sufficient but not necessary for (1) and (3) to hold*. The choice of $\phi(x)$ is however a best possible one in one sense. For we find that the conclusions (1) and (3) are false if we change the hypothesis so as to admit functions of logarithmico-exponential type only, for which

$$\int_a^\infty \frac{dx}{x\phi(x)}$$

is divergent. For in this case any given $\phi(x)$ must be less than $l_1(x)\ldots.l_k(x)$ for some integer k where k depends

* See my earlier paper, *loc. cit.*

† Valiron, *loc. cit.* p. 86.

on the order* of $\phi(x)$, and for all large x. Consider now

$$f_1(z)=\prod_N^{\infty}\left(1-\frac{z}{a_n}\right); f_2(z)=\prod_N^{\infty}\left\{\left(1-\frac{z}{b_n}\right)\exp\left(\frac{z}{b_n}\right)\right\},$$

where

$$a_n=-n(l_1n)\ldots(l_kn)(l_{k+1}n)^2,$$
$$b_n=n(l_1n)\ldots(l_{k+1}n);$$

$f_1(z)$ and $f_2(z)$ are canonical products of order one and of genus zero and one respectively. For each of them we have

$$\lim_{r\to\infty}\frac{\log M(r)}{n(r)l_1rl_2r\ldots l_kr}=\infty.$$

8. Finally we add that the result can be easily extended to meromorphic functions. Let $F(z)$ be a meromorphic function of order ρ. We can write

$$F(z)=z^k e^{g(z)}\frac{f_1(z)}{f_2(z)},$$

where $g(z)$ is a polynomial of degree q, and f_1 and f_2 are canonical products of genus p_1 and p_2 formed with the zeros and poles of $F(z)$ (other than the origin) respectively. Let $T(r, F)=T(r)$ be the Nevanlinna characteristic function for $F(z)$.

THEOREM 3. *If $F(z)$ be a meromorphic function of integral order $\rho \geqslant 1$, then*

$$\liminf_{r\to\infty}\frac{T(r)}{\{n(r,0)+n(r,\infty)\}\phi(r)}=0 \qquad (11)$$

except when $q>\mathrm{Max}\,(p_1, p_2)$.

The proof is straightforward.

* Hardy, *Orders of Infinity*, pp. 17—18.

NOTE ON A THEOREM OF POLYA.

BY

S. M. Shah, *Aligarh*.

[Received in revised form, 25 October 1941]

1. Let $f(z)$ be an integral function of order ρ. G. Polya* proved that

$$\varliminf_{r\to\infty} \frac{n(r)}{\log M(r)} \leqslant \rho \tag{1}$$

where $M(r)$ and $n(r)$ have their usual meanings. G. Valiron† showed that if $f(z)$ be of finite order

$$\varliminf_{r\to\infty} \frac{n(r)\log r}{\log M(r) \log\log M(r)} \leqslant 1. \tag{2}$$

We prove here

Theorem 1. *If $f(z)$ be an integral function and if*

$$\lambda_1 = \varliminf_{r\to\infty} \frac{\log^+ n(r)}{\log r}$$

then

$$\varliminf_{r\to\infty} \frac{n(r)}{\log M(r)} \leqslant \lambda_1. \tag{3}$$

Since $\lambda_1 \leqslant \lambda = \varliminf \{ \log\log M(r) \mid \log r \}$, (2) and (1) follow immediately from (3).

2. Lemma. *If $\chi(x)$ is a positive real function, continuous almost everywhere in (r_0, ∞) and if*

$$\varlimsup_{r\to\infty} \frac{\log \chi(x)}{\log x} = \alpha,$$

then

$$\varliminf_{r\to\infty} \frac{1}{\chi(r)} \int_{r_0}^{r} \frac{\chi(t)\,dt}{t} \geqslant \frac{1}{\alpha}.$$

* *Math. Annalen* 88 (1923), 169—83. Polya und Szego, Aufgaben II, P. 10, Ex. 63.

† *Bull. Sc. Math.* 48 (1924), 9—12.

Proof. The lemma is obviously true if α be infinite or if $\alpha<0$. Hence we need consider when α is finite and $\alpha\geqslant 0$. Suppose first $\alpha>0$, and suppose the lemma to be false. Let

$$\xi(r)=\int_{r_0}^{r}\frac{\chi(t)\,dt}{t}.$$

We have then $\xi(r)<c\chi(r)$ for all $r\geqslant\delta=\delta(c)>r_0$, where $c<1/\alpha$ is a positive constant. $\xi'(r)$ exists and is equal to $\chi(r)/r$ almost everywhere. Hence

$$\frac{\xi'(x)}{\xi(x)}>\frac{\chi(x)}{x}\,\frac{1}{c\chi(x)}=\frac{1}{cx}$$

almost everywhere in (δ,∞). Hence for all large r,

$$\log\,\xi(r)=\log\,\xi(\delta)+\int_{\delta}^{r}\frac{\xi'(x)}{\xi(x)}\,dx>\log\,\xi(\delta)+\frac{1}{c}\int_{\delta}^{r}\frac{dx}{x}$$

and hence

$$\log\,c+\log\,\chi(r)>\log\,\xi(r)>\log\,\xi(\delta)+\frac{1}{c}\log\frac{r}{\delta}.$$

Hence

$$\lim_{r\to\infty}\frac{\log\,\chi(r)}{\log r}\geqslant\frac{1}{c}>\alpha$$

which contradicts the hypothesis. Hence the lemma is proved. If $\alpha=0$ then we take c to be a large constant.

Proof of Theorem 1. We suppose that $n(r)$ tends to infinity with r, for otherwise the theorem is obvious. We may also suppose $f(0)\neq 0$. By Jensen's Theorem

$$\frac{\log M(r)}{n(r)}>\frac{1}{n(r)}\int_0^r\frac{n(t)\,dt}{t}+\frac{k}{n(r)}.$$

We apply the lemma putting $\chi(x)=n(x)$ and deduce that

$$\overline{\lim_{r\to\infty}}\frac{\log M(r)}{n(r)}\geqslant\overline{\lim_{r\to\infty}}\left\{\frac{1}{n(r)}\int_0^r\frac{n(t)\,dt}{t}\right\}\geqslant\frac{1}{\lambda_1}$$

and the theorem follows.

3. We can similarly prove

Theorem 2. *If $f(z)$ be a meromorphic function, and if*

$$\lambda_1(a) = \overline{\lim_{r\to\infty}} \frac{\log^+ n(r, a)}{\log r}; \ \mu = \overline{\lim_{r\to\infty}} \frac{\log T(r)}{\log r}$$

then

$$\underline{\lim_{r\to\infty}} \frac{n(r, a)}{T(r)} \leqslant \lambda_1(a) \leqslant \mu$$

for every a $(0 \leqslant |a| \leqslant \infty)$.

Cor. 1. *If $f(z)$ be an integral function, then*

$$\underline{\lim_{r\to\infty}} \frac{n(r, a)}{\log M(r)} \leqslant \lambda_1(a) \leqslant \lambda,$$

where a is any finite constant.

Cor. 2. *If $f(z)$ be a meromorphic function of zero order then*

$$\underline{\lim_{r\to\infty}} \frac{n(r, a_1) + \ldots\ldots + n(r, a_q)}{T(r)} = 0,$$

where the a's $(0 \leqslant |a_\nu| \leqslant \infty)$ are any constants.

This follows immediately if we put

$$\chi(x) = n(x, a_1) + \ldots + n(x, a_q)$$

in the lemma.

Cor. 3. *If $f(z)$ be an integral function of zero order then*

$$\underline{\lim_{r\to\infty}} \frac{n(r, a_1) + \ldots + n(r, a_q)}{\log M(r)} = 0$$

where the a's are any finite constants.

ON NUMBERS OF THE FORM $4^a(8b+7)$.

BY

HANSRAJ GUPTA, *Hoshiarpur*.

[Received 16 May 1941]

1. Let $k(n)$ denote the number of integers not exceeding n, which cannot be expressed as the sums of less than four squares. Then since all such numbers are of the form $4^a(8b+7)$, where a and b are non-negative integers, we have

$$k(n) = \sum_{a=0}^{\infty}\left[\frac{[n/4^a]+1}{8}\right]; \tag{1}$$

or

$$k(n)-k([n/4]) = [(n+1)/8]. \tag{2}$$

In this note, closely following Mr. M. C. Chakrabarti's method*, I obtain for $k(n)$ an expression very similar to the one given by him. My result is stronger however, in so far as the three theorems of Chakrabarti follow immediately from it. Besides this, my expression provides a quicker way of calculating $k(n)$ for any given value of n.

2. Let $n=(h_s h_{s-1} h_{s-2} \ldots h_2 h_1 h_0)_4$, where $h_s>0$, be the representation of n in the scale of 4. Then

$$k(n) = \sum_{i=0}^{s-1}\left[\frac{(h_s h_{s-1} \ldots . h_i)_4+1}{8}\right]. \tag{3}$$

Now for every value of $i>0$,

$$\sum_{a=0}^{i-1}\left[\frac{4^{i-a}h_i}{8}\right] = \frac{4^i h_i}{6} - \lambda, \tag{4}$$

where $\lambda = \frac{2}{3}, \frac{1}{3}$ or 1, according as $h_i = 1, 2$ or 3.

* M. C. Chakrabarti, On the limit points of a function connected with the Three-Square-Problem, *Bull. Calcutta Math. Soc.*, 32, (1940), 1—6.

Also

$$\left[\frac{(h_s h_{s-1}\ldots h_i)_4+1}{8}\right]=\left[\frac{(h_s h_{s-1}\ldots h_i)_4}{8}\right]+u_i,\ i\geqslant 0; \qquad (5)$$

where $u_i = 1$, if and only if $h_i = 3$ and h_{i+1} is odd,
and $= 0$ otherwise.

Hence it is easily shown that

$$k(n) = \tfrac{1}{6}n-\tfrac{2}{3}n_1-\tfrac{1}{3}n_2-n_3-\tfrac{1}{6}h_0+u_0; \qquad (6)$$

i.e.

$$n-6\,k(n) = 4n_1+2n_2+6n_3+h_0-6u_0; \qquad (7)$$

where n_1 and n_2 are respectively the numbers of 1's and 2's among the digits $h_s, h_{s-1}, \ldots, h_3, h_2, h_1$; and n_3 is the number of those non-zero positive values of i for which $h_i = 3$, and $h_{i+1} = 0$ or 2.

For purposes of this rule, we take $h_{s+1} = 0$, when $h_s = 3$ and $s \geqslant 1$.

3. For values of $n>0$, the right side of (7) cannot be less than 1, nor can it be greater than $4(s+1)$, i.e. $4\log_4(4n)$. In fact, we have the

Theorem. *If* $1 \leqslant m \leqslant 4(s+1)$, *where* $s \geqslant 1$, *then for some* n *in the interval* $4^s \leqslant n < 4^{s+1}$,

$$n-6\,k(n) = m. \qquad (8)$$

It is easily seen that the theorem holds when $s = 1$ or 2. Suppose the theorem holds also when $2 < s \leqslant (t-1)$. First, let $1 \leqslant m \leqslant 4t$. Suppose

$$n = n' \equiv (h_{t-1}h_{t-2}\ldots h_2h_1h_0)_4,$$

where $h_{t-1}>0$ is a solution of (8) in the interval $(4^{t-1}, 4^t)$. Then

$$n = (h_{t-1}h_{t-2}\ldots h_{i+1}h_th_i\ldots h_1h_0)_4,\ i>0;$$

where $h_t = 3$ or 0, according as h_i is or is not equal to 3, is a solution of (8) in the interval $(4^t, 4^{t+1})$. Secondly, let $m = 4t+j$, where $j = 1,2,3$ or 4. Suppose

$$n = n'' \equiv (h_{t-1}h_{t-2}\ldots h_2h_1h_0)_4,$$

where $h_{t-1}>0$, is a solution of the equation

$$n-6\,k(n) = m-4.$$

Now at least one of the digits $h_1, h_2, \ldots, h_{t-1}$ must be different from 3, for otherwise $n = n''$ would not satisfy the relation stated above. Hence we can suppose that $h_i \neq 3$, for some $i > 0$. Hence

$n = (h_{t-1} \ldots h_{i+1} h_t h_i \ldots h_1 h_0)_4$, where $h_t = 1$ is a solution of (8) in the interval $(4^t, 4^{t+1})$; and the theorem follows readily by induction.

4. We shall now deduce as corollaries of the above theorem, the three theorems of Chakrabarti which, in our notation, are the following :

THEOREM A. $\liminf \dfrac{n-6\,k(n)}{6 \log n} = 0.$

THEOREM B. $\limsup \dfrac{n-6\,k(n)}{6 \log n} = \dfrac{1}{\log 8}.$

THEOREM C. *Every point in the interval* $\left(0, \dfrac{1}{\log 8}\right)$ *is a limit point of the sequence* $\dfrac{n-6\,k(n)}{6 \log n}$.

We first notice that the equation

$$n - 6\,k(n) = m,$$

holds for an infinite number of values of n; for if $m = 4q + r$, where $1 \leqslant r \leqslant 4$, $q \geqslant 0$; then there is at least one solution of the equation in each of the intervals:

$$(4^q, 4^{q+1}),\ (4^{q+1}, 4^{q+2}),\ (4^{q+2}, 4^{q+3}),\ \ldots\ldots$$

Secondly, if n_j be any solution of the said equation, such that $4^j \leqslant n_j < 4^{j+1}$, then writing u_j for $\dfrac{n_j - 6\,k(n_j)}{6 \log n_j}$, we have

$$u_j \leqslant \frac{m}{6 \log 4^j} = \frac{m}{4j} \cdot \frac{1}{\log 8}.$$

Also
$$u_j \geqslant \frac{m}{6 \log 4^{j+1}} = \frac{m}{4(j+1)} \cdot \frac{1}{\log 8}.$$

Hence

$$\frac{m}{4j} \geqslant u_j \log 8 \geqslant \frac{m}{4(j+1)}.$$

Since $m \leqslant 4(j+1)$, therefore $m/4j > 1$ only when $m = 4j+r$, $1 \leqslant r \leqslant 4$. If ε be any preassigned positive quantity however small, then $m/4j \geqslant 1+\varepsilon$, only when $r/4j \geqslant \varepsilon$, i.e. $j \leqslant r/4\varepsilon$.

Thus there are only a finite number of values of j for which

$$u_j \log 8 \geqslant 1+\varepsilon.$$

Hence $\limsup_{j\to\infty} u_j = 1/\log 8$.

Again if $m/4j < \varepsilon$, i.e. $j > m/4\varepsilon$, then $u_j \log 8 < \varepsilon$.

Since u_j is necessarily positive, we get $\liminf_{j\to\infty} u_j = 0$.

Lastly, if t be any point in the range $0 \leqslant t \leqslant 1$, and we choose the integer m so that

$$4(j+1)(t-\varepsilon) \leqslant m \leqslant 4j\,(t+\varepsilon),$$

$\varepsilon < t$ being any preassigned positive quantity however small, then

$$t-\varepsilon \leqslant u_j \log 8 \leqslant t+\varepsilon.$$

An integer m evidently exists for all values of j for which

$$4j\,(t+\varepsilon) - 4(j+1)(t-\varepsilon) \geqslant 1,$$

i.e.

$$j \geqslant \frac{1+4(t-\varepsilon)}{8\varepsilon}.$$

Thus every point in the range $0 \leqslant t \leqslant 1$, is a limit point of $u_j \log 8$.

5. In the rest of this paper, we consider the following

PROBLEM. *Given m, for how many values of n, $0 \leqslant n < 4^s$, does equation* (8) *hold ?*

The non-negative integers $n < 4^s$, where $s \geqslant 1$, can be expressed in the scale of 4 in the form

$$(h_{s-1}h_{s-2}\ldots h_2h_1h_0)_4$$

where the h's are the digits 0, 1, 2 and 3.

We divide these integers into two classes, placing in the first class all those for which h_{s-1} is 0, 1 or 2, and in the second class those for which h_{s-1} is 3. It should be clearly noticed that the same n may belong to different classes for different values of s. Thus for $s=2$, the integer 15 belongs to class two; but for $s=3$, it belongs to class one; it being written as (33) in the former case and as (033) in the latter.

Let $N_1(m,s)$ denote the number of those solutions of (8), $0 \leqslant n < 4^s$, which belong to the first class and $N_2(m,s)$ the number of those that belong to the second class.

Let $n' \equiv (h_{s-1}h_{s-2}\ldots h_2h_1h_0)_4$ be any solution of (8), $0 \leqslant n < 4^s$. Consider the value of $n-6\,k(n)$, when

$$n = n'' \equiv (h_s h_{s-1} h_{s-2} \ldots h_2 h_1 h_0)_4.$$

Evidently,

$$n''-6\,k(n'') = m, \text{ when } h_s = 0;$$
$$= m+2, \text{ when } h_s = 2;$$
$$= (m-2) \text{ or } (m+4), \text{ when } h_s = 1,$$

according as h_{s-1} is or is not 3;

and is equal to m or $(m+6)$, when $h_s = 3$, according as h_{s-1} is or is not 3.

Hence we have the following recurrence formulae:

$$N_2(m,\ s+1) = N_2(m,s) + N_1(m-6,s); \tag{9}$$

and

$$N_1(m,\ s+1) = N_1(m,s) + N_2(m,s) + N_1(m-2,s) + N_2(m-2,s) + N_1(m-4,s) + N_2(m+2,\ s). \tag{10}$$

Starting with the initial values

$$N_1(m,1) = 1, \text{ if and only if } m = 0, 1 \text{ or } 2;$$

and $= 0$ otherwise;

while $N_2(m,1) = 1$, if and only if $m = 3$,

and $= 0$ otherwise;

we can calculate in succession the values of $N_1(m,s)$ and $N_2(m,s)$ with the help of the recurrence formulae stated

above. The total number of solutions $N(m, s)$ of (8) is then given by the relation

$$N(m, s) = N_1(m, s) + N_2(m, s). \tag{11}$$

6. Before proceeding to obtain some general results, we calculate the values of $N_1(m, s)$ and $N_2(m, s)$ for values of $s \leqslant 5$. The following scheme is very helpful in the calculation work. First, when m is even, we have

$s \rightarrow$	1		2		3		4		5	
$m \downarrow$	N_1	N_2	N_1	N_2	N_1	N_2	N_1	N_2	N_1	N_2
0	1	0	1	0	1	0	1	0	1	0
2	1	0	2	0	3	0	4	0	5	0
4	0	0	2	0	6	0	12	0	20	0
6	0	0	1	1	7	2	21	3	46	4
8	0	0	0	1	5	3	25	6	75	10
10	0	0	0	0	2	2	20	8	88	20
12	0	0	0	0	0	1	10	8	76	29
14	0	0	0	0	0	0	3	5	48	30
16	0	0	0	0	0	0	0	2	20	22
18	0	0	0	0	0	0	0	0	5	10
20	0	0	0	0	0	0	0	0	0	3

$$48 = (20+10+3)+(8+5+2).$$

Secondly, when m is odd, the same scheme gives

$s \rightarrow$	1		2		3		4		5	
$m \downarrow$	N_1	N_2	N_1	N_2	N_1	N_2	N_1	N_2	N_1	N_2
1	1	0	2	0	3	0	4	0	5	0
3	0	1	2	1	5	1	9	1	14	1
5	0	0	2	0	8	0	20	0	40	0
7	0	0	0	1	5	3	23	6	65	10
9	0	0	0	0	3	2	23	7	89	16
11	0	0	0	0	0	2	12	10	80	30
13	0	0	0	0	0	0	5	5	58	28
15	0	0	0	0	0	0	0	3	25	26
17	0	0	0	0	0	0	0	0	8	12
19	0	0	0	0	0	0	0	0	0	5

$$30 = 20+10.$$

7. We can write (9) in the form

$$\mathcal{N}_2(m,\ s)-\mathcal{N}_2(m,\ 1)=\sum_{i=1}^{s-1}\mathcal{N}_1(m-6,\ i). \tag{12}$$

For values of $m<6$, the right side of (12) is zero.

Hence $\mathcal{N}_2(m,\ s)=\mathcal{N}_2(m,\ 1),\ m<6$;

i.e. $\mathcal{N}_2(m,\ s)=0$, when $m=0,\ 1,\ 2,\ 4$ or 5;

and $=1$, when $m=3$.

For values of $m\geqslant 6$, $\mathcal{N}_2(m,\ 1)=0$;

therefore
$$\mathcal{N}_2(m,\ s)=\sum_{i=1}^{s-1}\mathcal{N}_1(m-6,\ i),\ m\geqslant 6. \tag{13}$$

Again, we can write (10) in the form

$$\mathcal{N}_1(m,\ s)-\mathcal{N}_1(m,\ 1)=\sum_{i=1}^{s-1}\{\mathcal{N}_1(m-2,\ i)+\mathcal{N}_1(m-4,\ i)$$
$$+\mathcal{N}_2(m+2,\ i)+\mathcal{N}_2(m,\ i)+\mathcal{N}_2(m-2,\ i)\}. \tag{14}$$

Giving to m the values 0, 1, 2 and 3 in succession, we get

$$\mathcal{N}_1(0,\ s)=1=\binom{s-1}{0};\ \mathcal{N}_1(1,s)=\binom{s}{1}=\mathcal{N}_1(2,\ s),$$

and
$$\mathcal{N}_1(3,\ s)=\tfrac{1}{2}(s-1)(s+2)=\binom{s}{2}+\binom{s-1}{1}.$$

From (13), we now obtain

$$\mathcal{N}_2(6,\ s)=\binom{s-1}{1};\quad \mathcal{N}_2(7,\ s)=\binom{s}{2}=\mathcal{N}_2(8,\ s),$$

and
$$\mathcal{N}_2(9,\ s)=\binom{s}{3}+\binom{s-1}{2}.$$

Proceeding in this manner, it can be shown that $\mathcal{N}_1(m,\ s)$ is a rational integral polynomial in s of degree $\left[\frac{m+1}{2}\right]$; while $\mathcal{N}_2(m,\ s)$ is a rational integral polynomial in s of degree $\left[\frac{m-3}{2}\right]$, $m\geqslant 6$.

Moreover, $\mathcal{N}_1(m,\ s)$ vanishes for positive integral values of $s<(m+2)/4$, while $\mathcal{N}_2(m,\ s)=0$ for values of $s<m/4$.

Hence some of the algebraic factors of these polynomials for $N_1(m, s)$ and $N_2(m, s)$ can always be readily determined.

Thus $N_1(8, s) = 0$ when $s = 1$ and also when $s = 2$.

Therefore $N_1(8, s) = (s-1)(s-2)(a_0 s^2 + a_1 s + a_2)$, where the a's are certain constants, not necessarily integers. Since $N_1(8, 3) = 5$, $N_1(8, 4) = 25$, and $N_1(8, 5) = 75$, we get $a_0 = \frac{5}{24} = a_1$, and $a_2 = 0$. Thus

$$N_1(8, s) = 5\binom{s+1}{4}.$$

In the same manner, we can show that

$$N_1(4, s) = 2\binom{s}{2},\ N_1(5, s) = 2\binom{s+1}{3},$$

$$N_1(6, s) = \tfrac{1}{2}(s-1)(s^2-2) = 3\binom{s}{3} + \binom{s}{2} + \binom{s-1}{2},$$

and $N_1(7, s) = 3\binom{s+1}{4} + 2\binom{s}{3}$.

8. Since $N_2(m, s) = \sum_{i=1}^{s-1} N_1(m-6, i), \quad m \geqslant 6;$

we can express (14) in the form

$$N_1(m, s) = \sum_{i=1}^{s-1} \{ N_1(m-2, i) + N_1(m-4, i) \}$$

$$+ \sum_{i=1}^{s-1} \sum_{j=1}^{i-1} \{ N_1(m-4, j) + N_1(m-6, j) + N_1(m-8, j) \}, m \geqslant 8. \quad (15)$$

These formulae provide a very quick method of obtaining expressions for $N_1(m, s)$ and $N_2(m, s)$, when $N_1(m-2, s)$, $N_1(m-4, s)$, $N_1(m-6, s)$ and $N_1(m-8, s)$ have been expressed as sums of combinatory functions.

We here introduce the operator ∇ defined by the relations:

$$\nabla\binom{j}{r} = \binom{j}{r+1},\quad \nabla^i\binom{j}{r} = \binom{j}{r+i}.$$

This enables us to write (13) and (15) in the form:

$$N_2(m, s) = \nabla N_1(m-6, s),\ m \geqslant 6. \quad (16)$$

$$\mathcal{N}_1(m, s) = \nabla \{ \mathcal{N}_1(m-2, s) + \mathcal{N}_1(m-4, s) \} + \nabla^2 \{ \sum_{j=2}^{4} \mathcal{N}_1(m-2j, s) \},\ m \geqslant 8. \tag{17}$$

Hence, we have for values of $m \geqslant 8$,

$$\mathcal{N}(m, s) = \nabla \{ \sum_{j=1}^{3} \mathcal{N}_1(m-2j, s) \} + \nabla^2 \{ \sum_{j=2}^{4} \mathcal{N}_1(m-2j, s) \}. \tag{18}$$

Thus $\mathcal{N}_1(9, s) = \nabla \{ \mathcal{N}_1(7, s) + \mathcal{N}_1(5, s) \} + \nabla^2 \{ \mathcal{N}_1(5, s) + \mathcal{N}_1(3, s) + \mathcal{N}_1(1, s) \},$

$$= 3\binom{s+1}{5} + 2\binom{s}{4} + 2\binom{s+1}{4} + 2\binom{s+1}{5} + \binom{s}{4} + \binom{s-1}{3} + \binom{s}{3},$$

$$= 3\binom{s+2}{5} + 2\binom{s+1}{5} + 2\binom{s}{4} + \binom{s-1}{3}.$$

9. In what follows, we consider two interesting cases of the above problem, and show that when $m = 4s-1$, the number of solutions of (8) is given by the sth term in the series of Fibonacci viz. the series:

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

in which each term is the sum of the preceding two; and when $m=4s$, the number of solutions is given by the $(s-1)$th term of the same series.

First, when $m = 4s-1$, the 'key'-solution is given by

$$n = (31^{s-1})_4$$

where the index attached to the digit 1 indicates the number of times it occurs in the representation. Other solutions can be constructed from the key-solution by replacing any digit-pair (11) by the digit-pair (23). Thus when $s = 5$, the solutions are:

$$n = (31111)_4,\ (32311)_4,\ (31231)_4,\ (31123)_4,\ (32323)_4.$$

The number of solutions L_s is thus the same as the number of numbers of $(s-1)$ digits which can be formed

with 1's and the digit-pairs (23). If all such numbers be written out, the number of numbers which have the digit 1 to their extreme right is L_{s-1}, and the number of those that have the digit-pair (23) to their extreme right is L_{s-2}.

Since these are the only possible cases that can arise, we must have

$$L_s = L_{s-1} + L_{s-2},$$

and the result follows by induction. Again, since the number of numbers that can be formed with $s-(2t+1)$ 'ones' and t digit-pairs (23)

$$= \binom{s-t-1}{t},$$

we get
$$L_s = \sum_{t=0}^{[(s-1)/2]} \binom{s-t-1}{t}.$$

This is another expression for the sth term in the series of Fibonacci. As an example, we have

$$L_5 = \binom{4}{0} + \binom{3}{1} + \binom{2}{2} = 5.$$

Secondly, when $m = 4s$, the 'key'-solution is

$$n = (31^{s-2}2)_4,$$

and proceeding as before, it can be shown that the number of solutions is given by the $(s-1)$th term in the series of Fibonacci. It may be noticed that $m=1$ only when $n=(13^j)_4, j \geqslant 0$; and $m=2$ only when $n=(20^j)_4$, $j \geqslant 0$.

10. The following is a graphical method for the determination of the number of solutions of the equation (8) when $4^s \leqslant n < 4^{s+1}$, $s \geqslant 1$.

Let $a_1, a_2, a_3, \ldots, a_\mu$, where $\mu = 2.\ 4^{s-1}$, be the numbers between 4^s and 4^{s+1} which cannot be expressed as the sums of less than four squares.

Plot the points : $(1, a_1 - 4^s - 2)$, $(2, a_2 - 4^s - 8)$, $(3, a_3 - 4^s - 14), \ldots\ldots\ldots, (\mu, a_\mu - 4^s + 4 - 6\mu)$.

Through these points draw lines each 5 units in length and parallel to the axis of y. Thus the jth line will extend from the point (j, a_j-4^s+4-6j), to the point (j, a_j-4^s+9-6j).

Then the number of solutions required is the same as the number of points in which the line $y=m$ meets the set of μ lines. The result easily follows from the fact that

$$\{n-6k(n)\}-\{(n-1)-6k(n-1)\}=-5 \text{ or } 1,$$

according as n is or is not of the form $4^a(8b+7)$.

Since none of the lines $y=m$ will in general intersect all the μ lines, the number of solutions for no m can exceed 2.4^{s-1}. In fact, except when $s=1$, the maximum number of solutions is always less. Hence,

If among the numbers,

$$a_j-4^s+4-6j,\ 1\leqslant j\leqslant 2.4^{s-1},$$

there be q_1 *'ones',* q_2 *'twos',* q_3 *'threes', and so on, then the number of solutions of* (8), $4^s\leqslant n<4^{s+1}$, $s\geqslant 1$ *is given by* $\sum_{i=0}^{5} q_{m-i}$.

ERRATA.

Page 116, line 16. for $\log\frac{a_n d_{n+1}}{a_{n+1}d_n}$ read $\log\frac{a_{n+1}d_n}{a_n d_{n+1}}$

Page 136, line 4 from the bottom.

Read: The constant is 1, since G (0,0) = 1. Equations (10), (11) and (13) have been

Vol. III (1939), pp. 268-70. The result of § 3 can be proved by putting $\alpha = n + \nu$ in Sonine's formula, and proceeding as in § 2.

THE JOURNAL

OF THE

INDIAN MATHEMATICAL SOCIETY

Editorial Committee:

R. VAIDYANATHASWAMI (*Chairman*)

K. ANANDA RAO
D. D. KOSAMBI
F. W. LEVI
S. S. PILLAI
C. RACINE
N. R. SEN
M. R. SIDDIQUI
T. VIJAYARAGHAVAN

C. N. SRINIVASIENGAR (*Secretary*)

NEW SERIES

Vol. V

1941

PRINTED AT THE MADRAS LAW JOURNAL PRESS, MYLAPORE, MADRAS.

Annual Subscription: Rs. 6.

CONTENTS

PAGES.

Chandrasekharan, K. On Hadamard's Factorization Theorem .. 128—132

Cherian, K. C. A Bivariate correlated Gamma-Type Distribution Function .. 133—144

Ganapathy Iyer, V. On periodic integral functions . 1—17

Hans Raj Gupta. Some properties of Generalized combinatory functions .. 27—31

——— On Numbers of the Form $4^a(8b+7)$ .. 192—202

Haridas Bagchi. On Bicircular Quartics .. 46—47

Hari Shanker. A Note on certain Self-reciprocal Functions .. 44—45

——— On Functions which are Hankel-Transforms of each Other .. 109—112

Kesava Menon, P. A Generalization of Legendre Functions .. 92—102

Kosambi, D. D. Path Equations admitting the Lorentz Group II .. 62—72

Levi, F. W. On the Number of Generators of a free product, and a Lemma of Alexander Kurosch. 149—155

Minakshisundaram, S. On the Expansion of an arbitrary Function in a Series of Eigen-functions of Boundary Value Problems II .. 103—108

Mohan, B. Self-Reciprocal Functions involving Appell's Function .. 32—37

——— Some Infinite Integrals .. 123—127

Narasimha Murthi, V. On Waring's Problem for 8th, 9th and 10th Powers .. 122

Pankajam, S (Miss). On the Formal Structure of the Propositional Calculus I .. 49—61

Racine, C. Contribution to the Relativistic Problem of n Bodies (I) .. 156—164

———Contribution to the Relativistic Problem of n Bodies (II) .. 165—178

Rajagopal, C. T. Postscript to "Convergence Theorems for Series of Positive Terms" .. 113—116

Ratnam, P. The Derivatives of an Algebraic Function .. 145—147

Sambasiva Rao, K. On Waring's Problem for Smaller Powers .. 117—121

Satish Chandra Chakrabarti. On a Few Recurrents 18—26

		Pages.
Shah, S. M. A Theorem on Integral Functions of integral order	..	179—188
—— Note on a Theorem of Polya	..	189—191
Srinivasiengar, C. N. The Linear Line-Congruence	..	73—91
Venkataraman, B. R. On a New Chain of Theorems in Circle-Geometry	..	38—43

129113
600 008.

JL
Bm2, N34
N34.5.4

Publications of the Indian Mathematical Society

1. Memoir on Cubic Transformations Associated with a Desmic System, by Dr. R. Vaidyanathaswami, pp. 92, Price Rs. 3.
2. Tables of Partitions, by Dr. Hansraj Gupta, pp. 81, Price Rs. 5.

Back Numbers of the *Journal* and of the *Student* are available, except for a few numbers out of stock.

THE MATHEMATICAL ASSOCIATION OF AMERICA

The Mathematical Association of America is devoted primarily to the interests of collegiate mathematics. It now has over two thousand individual and institutional members. There are twenty-two sections at present, representing thirty-one different States. The Association holds two national meetings per year and the sections hold one or two meetings per year. All meetings, both national and sectional, are reported in the official Journal.

Application blanks for membership may be obtained from the Secretary, Prof. W.D. Cairns, Oberlin, Ohio.

THE AMERICAN MATHEMATICAL MONTHLY

Official Organ of

THE MATHEMATICAL ASSOCIATION OF AMERICA

Is the Only Journal of Collegiate Grade in the Mathematical Field in this Country.

Most of its mathematical contributions can be read and understood by those who have not specialized in mathematics beyond the Calculus.

The historical papers, which are numerous and of high grade, are based upon original research.

The QUESTIONS and DISCUSSIONS, which are timely and interesting, cover a wide variety of topics.

Surveys of the contents of recent books constitute a valuable guide to current mathematical literature.

The topics in the department of Undergraduate Mathematics Clubs have excited wide interest among those engaged in teaching mathematics.

The NEWS and NOTICES cover a wide range of interest and information.

The Problems and Solutions hold the attention and activity of a large number of persons who are lovers of Mathematics for its own sake.

PRINTED BY R. NARAYANASWAMI IYER AT THE MADRAS LAW JOURNAL PRESS, MUNDAKANNI AMMAN KOIL STREET, MYLAPORE, MADRAS.